Strongly symmetric spectral convex sets are Jordan algebra state spaces

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Operator Algebras and Groups with Applications in Quantum Information, Universidad Autónoma de Madrid, June 20, 2019

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Research program: study information processing and physics in general probabilistic theories ("GPTs")

What?

Characterize quantum and classical theories within broad framework of "foil theories"...

...perhaps in terms of flow and processing of information. Understand how possibility/impossibility of information-processing protocols or physical phenomena (e.g. thermodynamic behavior) are connected with properties (geometric, etc...) of spaces of states, measurements, dynamics.

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Research program: study information processing and physics in general probabilistic theories ("GPTs")

Why?

From pragmatism...

- Conceptual understanding of info processing: principles \leftrightarrow tasks
 - ...help develop protocols for, or understand limits to, QIP ...
 - ...model info in other complex / concurrent systems?

...to hubris

- Information the essence of quantum physics? ...analogue of Einstein's principle-based account of special and general relativity? Wheeler's It from Bit?
- framework for possible new physics?

...and last but not least

Source of compelling mathematical questions about geometry of convex sets.

Main result

A result of pure convex geometry, but motivated by the interpretation of convex compact set as state space of an (e.g. physical) system.

Theorem (H. Barnum and Joachim Hilgert, arxiv:1904.03753)

Any finite-dimensional compact convex set that is (1) spectral and (2) strongly symmetric is the space of normalized states of a simple Euclidean Jordan algebra, or a simplex.

(Barnum, Mueller, Ududec ("BMU") *New J. Phys* **16** 123029 (2014), also arxiv:1403.4147) already observed the converse (easy from Jordan spectral theorem and transitivity of Jordan automorphisms on Jordan frames). BMU had characterized the same set of state spaces using the additional assumption of "no higher-order interference" in the sense of Rafael Sorkin (as adapted to GPTs by Ududec, Barnum & Emerson).

State spaces and automorphisms

Normalized states of system: Convex compact set $\Omega \subset A$ (affine space) of dimension *n*.

A state (element of Ω) is **pure** if it is extremal in the convex set Ω (i.e., not a nontrivial convex combination of distinct states in Ω). The pure states are the **extreme boundary** of Ω , written $\partial_e \Omega$.

Automorphism group of state space Ω (symmetries, candidate reversible dynamics): Group Aut Ω of affine maps $g : A \to A$ such that $g(\Omega) = \Omega$.

If we make *A* into a vector space by taking the centroid $c(\Omega)$ as 0, we can introduce an inner product making it into a Euclidean space *E* such that Aut $\Omega < O(E)$.

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Effects and measurements

Effect: affine functional: $e : A \to \mathbb{R}$ such that $e(\Omega) \subseteq [0, 1]$. Associated with a measurement outcome whose probability on $\omega \in \Omega$ is $e(\omega)$. **Unit effect:** constant functional u with u(A) = 1. **Measurements:** Indexed sets of effects e_i with $\sum_i e_i = u$ (more generally, effect-valued measures).

Give the convex compact set of effects *S* the **pointwise ordering**: $u \ge v := \forall x \in \Omega, \ u(x) \ge v(x)$. Then *S* is the order interval [0, u].

GPT may specify "allowed" or "physical" set of effects $\mathscr{E} \subseteq [0, u]$. **No-restriction** property holds if $\mathscr{E} = [0, u]$.

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Perfectly distinguishable states and frames:

Crucial notion used in formulating both strong symmetry and spectrality:

Definition (BMU 2014)

A sequence of pure states $\omega_1, ..., \omega_n \in \Omega$ is **perfectly distinguishable** if there exist allowed effects $e_1, ..., e_n \in \mathscr{E}$, with $\sum_i e_i \leq u$, such that $e_i(\omega_j) = \delta_{ij}$. We also call it a **frame** or an **n-frame**. It is **maximal** if it is not a proper subsequence of any other frame.

- order matters: different orders, different frames.
- Fewer effects, less distinguishability, so set of frames depends on both Ω and &, but when & = [0, u], depends only on Ω. In BH2019 we define distinguishability, frames using [0, u] so spectrality, strong symmetry are properties of Ω.

Spectrality and strong symmetry

- **Spectrality**: Every state $\omega \in \Omega$ is in the convex hull of a frame $\omega_1, ..., \omega_r$: $\omega = \sum_{i=1}^r p_i \omega_i, \sum_i p_i = 1, p_i \ge 0.$
- **Strong Symmetry**: Aut Ω acts transitively on the set of *n*-frames.

These abstract (1) the finite-dimensional spectral theorem for quantum statesd: every density matrix can be written $\rho = \sum_i \rho_i \pi_i$ where π_i are an orthonormal set of rank-1 Hermitian projectors (i.e. $v_i v_i^*$ for some orthonormal set of states v_i .) and (2) the fact that for two orthonormal sets π_i , σ_i with $i \in \{1, ..., m\}$, there is a unitary U such that for all i, $\sigma_i = U \pi_i U^{\dagger}$.

SSS \implies **Unique Spectrality:** p_i are unique up to a permutation.

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Jordan Algebra State Spaces

(Pascual Jordan, Nachr. Akad. Wiss. Göttingen 1933) **Jordan algebra**: real vector space *V* equipped with a commutative bilinear product $\bullet: V \times V \rightarrow V$) that satisfies a special case of associativity, the *Jordan property*: $a^2 \bullet (a \bullet b) = a \bullet (a^2 \bullet b)$, where $a^2 := a \bullet a$.

Abstracts properties from symmetric product $(A, B) \mapsto (AB + BA)/2$ on Hermitian matrices.

It's **Euclidean** (an EJA) if it is possible to introduce an inner product $(\cdot, \cdot) : V \times V \to \mathbb{R}$ such that $(a \bullet b, c) = (a, b \bullet c)$. In finite dim, equivalent to **formal reality**: $a^2 + b^2 = 0 \implies a = b = 0$.

Euclidean \implies has unit, *e*. Define tr x := (e, x).

The **normalized state space** of an EJA is the compact convex set $\Omega := \{x \in V : \text{tr } x = 1\}.$

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Jordan, von Neumann, Wigner classification

Ann. Math. 35, 29-34 (1934)):

the simple finite-dimensional formally real Jordan algebras are:

- Self-adjoint $n \times n$ matrices over \mathbb{R} , over \mathbb{C} , and over \mathbb{H} , with $A \bullet B := (AB + BA)/2$; V_+ is the positive semidefinite (PSD) ones; Ω the unit-trace ones.
- Spin factors: ℝ^p ⊕ ℝ with product
 (x, s) (y, t) = (tx + sy, ⟨x, y⟩ + st); V₊ Lorentz cone in p "space",
 one "time" dimension; Ω a Euclidean p-ball.
- **Exceptional** EJA: 3×3 self-adjoint octonionic matrices, $A \bullet B := (AB + BA)/2, V_+ = PSD, \Omega = PSD$ unit-trace.

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Linearization and the unnormalized state space

EJAs exemplify a general construction: embed $\Omega V \simeq \mathbb{R}^{n+1}$ as the base of a regular cone $V_+ := \mathbb{R}_+ \Omega$ of unnormalized states by embedding *A* (or its Euclideanization *E*) as an affine hyperplane in $V \setminus \{0\}$.

Cone in *V*: subset closed under nonnegative linear combinations. **regular** := pointed (contains no affine line), spans *V*, topologically closed.)

Aut $\Omega \curvearrowright E$ extends to a representation of Aut Ω as a subgroup of O(V), fixing $\mathbb{R}c(\Omega)$ pointwise.

V an ordered linear space: $x \ge y := x - y \in V_+$.

Effects extend to linear functionals $V \to \mathbb{R}$. [0, u] generates the **dual cone** V_+^* , of functionals nonnegative on V_+ . $u \in \operatorname{int} V^{*+}$ and $[0, u] = V^{*+} \cap (u - V^{*+})$

Euclideanity of a Jordan algebra makes the cone of squares regular, hence a candidate for unnormalized state space: suitable affine slices (e.g. $\Omega := \{x \in V : \text{tr } x = 1\} \cap V_+$) are compact.

More information on simple EJAs

Notation: $\Omega := \{x \in V_+ : \text{tr } x = 1\}$. $\mathfrak{lieAut} V_+ =: \mathbb{R} \oplus \mathfrak{g}, K := \text{Aut } \Omega, \mathfrak{k} := \mathfrak{lieAut } \Omega.$

Facts: Aut $\Omega = \text{Aut } V_+ \cap O(V)$, \mathfrak{g} semisimple with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Aut $V_+ \frown V$ iso to Aut $V_+ \frown \mathfrak{p}$ corestricted from the adjoint representation.

Table: Simple Euclidean Jordan algebras with associated cones and Lie algebras

V	V_+	g	ŧ	dim V	rank V
$Sym(m,\mathbb{R})$	$PSD(m,\mathbb{R})$	$\mathfrak{sl}(m,\mathbb{R})\oplus\mathbb{R}$	o(<i>m</i>)	$\frac{m(m+1)}{2}$	т
$Herm(m, \mathbb{C})$	$PSD(m,\mathbb{C})$	$\mathfrak{sl}(m,\mathbb{C})\oplus\mathbb{R}$	$\mathfrak{su}(m)$	$\bar{m^2}$	т
$Herm(m, \mathbb{H})$	$PSD(m, \mathbb{H})$	$\mathfrak{sl}(m,\mathbb{H})\oplus\mathbb{R}$	$\mathfrak{su}(m,\mathbb{H})$	<i>m</i> (2 <i>m</i> -1)	т
$\mathbb{R} \oplus \mathbb{R}^{n-1}$	Lorentz $(1, n-1)$	o(1, <i>n</i> −1)	o(<i>n</i>)	п	2
$\mathit{Herm}(3,\mathbb{O})$	<i>PSD</i> (3, 0)	¢6(-26)	f4	27	3

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Reducibility of systems

- A cone V_+ in a vector space V is **reducible** if $V = V_1 \oplus V_2$ and every extremal ray of V_{+} is either in V_{1} , or in V_{2} . We write $V_{+} = V_{1+} \oplus V_{2+}$.
- V_1 and V_2 are "superselection sectors". "No coherence between them". "Which sector?" is "classical information".
- Every (f.d.) regular cone decomposes uniquely as $V_{+} = \bigoplus_{i} V_{i+}$.
- For EJA state spaces, simplicity of the EJA \equiv irreducibility of V_{+} .
- For an "*n*-state" classical system $\Omega = \triangle_{n-1}, V_+ = \bigoplus_{i=1}^n \mathbb{R}_+$

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Outline of the proof

- Barnum/Müller/Ududec 2014: Ω SSS ⇒ every face of Ω is generated by a frame. If *F* ≤ *G*, a frame for *F* extends to one for *G*. All frames for *F* have same size, called the **rank** of *F*.
- A flag of a compact convex set Ω is a sequence F₁ ⊊ F₂ ⊊ ··· F_r of nonempty faces. Maximal if not a subsequence of another flag.
- Farran/Robertson 1994: Ω is regular := Aut Ω acts transitively on the set of maximal flags.
- Barnum/Hilgert 2019: Ω SSS ⇒ maximal frames ω₁,..., ω_r in bijection with maximal flags F₁,..., F_r via F_i = Vⁱ_{j=1}{ω_i}. Transitive action on max flags ⇒ transitive action on max frames (i.e. SSS ⇒ regular).

Farran/Robertson 1994: given max flag Φ = F₁,..., F_r of regular Ω, define L_Φ(≤ E) = Aff {c(F₁),..., c(F_r)}, π_Φ(Ω) = L_Φ ∩ Ω. L_Φ is a regular polytope (with automorphism group K_L/K^L (aka N_K(L)/Z_K(L)) and (obviously) independent of Φ.
 Ω = Aut Ω.π(Ω) = Aut Ω.ν where v is any vertex of π(Ω).

Outline of proof, continued

- Farran/Robertson 1994, Madden/Robertson 1995: Let Ω be regular, F be a face of $\pi(\Omega)$ and $K = \operatorname{Aut} \Omega$, and write $K^{c(F)}$ ($\simeq N_{K}(F)/Z_{K}(F)$) for the stabilizer of c(F) in K. Then $K^{c(F)}$. F is a face of Ω , call it H_{F} , and each face of Ω is $k.H_{F}$ for some face F of $\pi(B)$ and some $k \in K$.
- Barnum/Hilgert 2019: Ω SSS ⇒ π(Ω) is a simplex whose vertices are a maximal frame. (Uses further structure theory of SSS sets from BMU 2014, and preceding item.)
- Dadok 1985: A representation of a compact connected group ≤ O(V) (V real, f.d.) is called **polar** if there is a **Cartan subspace**: one meeting every orbit orthogonally.
- Madden/Robertson 1995: Ω regular ⇒ Aut Ω ∩ V irreducible polar, L_Φ a Cartan subspace.
- Dadok 1985: Irreducible polar representations are orbit-equivalent to isotropy representations K ∩ p of irreducible symmetric spaces G/K (G is semisimple, g = t+p the Cartan decomposition of g = litG, K ∩ p obtained from ad G by restriction & corestriction).

Madden-Robertson classification of regular convex bodies Theorem

Let G/K be an irreducible noncompact symmetric space, the compact group K connected, K ∩ p the isotropy representation, i.e. g = t+p is the Cartan decomposition of g := lit G and K ∩ p is the restriction to K of the corestriction to p of the adjoint action G ∩ g. Let a be a maximal abelian subspace of p and let v ∈ a be such that P := Conv W_K.v ⊂ a is a regular polytope, where W_K := K_a/K^a is the Weyl group of K. Then B := K.P = Conv K.v is a nonpolytopal regular convex body, P = π(B), and P = a ∩ B. B is determined, up to affine isomorphism, by the affine isomorphism class of π(B) and the symmetric space.

Conversely, for every nonpolytopal regular convex body B there is an irreducible noncompact symmetric space G/K such that B = Conv K.v in the isotropy representation K ~ p of compact connected K. π(B) is α∩ B, and is a W_K-orbitope.

Madden-Robertson classification, continued

Theorem (continued)

- In the above, v can be any strictly positive multiple of any fundamental weight w for which Conv W_K.w is a regular polytope. Fundamental weights are specified by marking a node of the Coxeter diagram, and Coxeter indicated which marked nodes give weights such that Conv W_K.w is a regular polytope.
- The list of irreducible noncompact symmetric spaces, with their dimensions, ranks, and the regular polytopes that occur as Weyl orbitopes in Cartan subspaces of their isotropy representations, is given in Tables 2, 3 and 4, which except for their last column are essentially Tables 2,3, and 4 of Madden/Robertson.

Madden-Robertson classification: tables

Table: Classical noncompact symmetric spaces with associated isotropy representations and Farran-Robertson polytopes (adapted from Madden/Robertson, Table 2)

		Rank of symmetric	Dimension of isotropy			
Туре	Symmetric space G/K	space	representation	Root space	Polytope	EJA
AI	$SL(n,\mathbb{R})/SO(n)$	<i>n</i> – 1	(n-1)(n+2)/2	A_{n-1}	\triangle_{n-1}	Herm(n,ℝ)
All	$SU^*(2n)/Sp(n)$	<i>n</i> – 1	(n-1)(2n+1)	A_{n-1}	\triangle_{n-1}	Herm(n,⊞)
AIII	$SU(p,q)/S(U_p \times U_q)$	q	2pq	$\begin{cases} C_q & (q < p) \\ B_q & (q = p) \end{cases}$	\Box_q, \Diamond_q	$\mathbb{R}^2 \oplus \mathbb{R} \ (p = q = 1)$
BI	$SO_0(p,q)/(SO(p) imes SO(q)) \ p+q ext{ odd}, \ q < p$	q	pq	Bq	\Box_q, \Diamond_q	$\mathbb{R}^{p} \oplus \mathbb{R} \ (q=1)$
DI	$SO_0(p,q)/(SO(p) imes SO(q)) \ p+q$ even	q	pq	$\left\{ egin{array}{ll} B_q & (q < p) \ D_q & (q = p) \end{array} ight.$	\Diamond_q	$\mathbb{R}^{p}\oplus\mathbb{R} \ (q=1)$
DIII	$SO^*(2n)/U(n)$	$\lfloor n/2 \rfloor = q$	<i>n</i> (<i>n</i> -1)	$\begin{cases} C_q & q \text{ odd} \\ BC_q & q \text{ even} \end{cases}$	\Box_q, \Diamond_q	$\mathbb{R}^2 \oplus \mathbb{R} \ (q=1)$
CI	$Sp(n,\mathbb{R})/U(n)$	n = q	n(n+1)	C_q	\Box_q, \Diamond_q	$\mathbb{R}^2 \oplus \mathbb{R} \ (q=1)$
CII	Sp(p,q)/(Sp(p) imes Sp(q))	q	4 <i>pq</i>	$\begin{cases} C_q & (q=p) \\ BC_q & (q < p) \end{cases}$	\Box_q, \Diamond_q	$\mathbb{R}^4 \oplus \mathbb{R} \ (p = q = 1)$

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Table: Exceptional noncompact symmetric spaces with associated isotropy representations and Farran-Robertson polytopes (adapted from Madden/Robertson, Table 3)

Туре	Symmetr presen	ic space G/K ted by g, \mathfrak{k}	Rank of symmetric space	Dimension of isotropy representation	Root space	Polytope	EJA
EIII	€6(-14)	$\mathfrak{so}(10) \oplus \mathbb{R}$	2	32	B ₂		
EIV	€6(-26)	f4	2	26	A ₂	∆ ₂	<i>Herm</i> (3, ^ℚ)
EVI	e7(-5)	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	4	64	F ₄	24-cell	
EVII	¢7(-25)	$\mathfrak{e}_6 \oplus \mathbb{R}$	3	54	C_3	□3,◊3	
EIX	e ₈₍₋₂₄₎	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	4	112	F ₄	24-cell	
FI	f4(4)	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	4	28	F ₄	24-cell	
FII	f4(-20)	so(9)	1	16	A ₁	\triangle_1	
G	\$2(2)	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	2	8	G ₂	hexagon	

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Table: Noncompact symmetric spaces arising as $K^{\mathbb{C}}/K$ for simple K, with associated isotropy representations and Farran-Robertson polytopes (Madden/Robertson, Table 4)

Type and root space	Symmetric space	Dimension of isotropy representation	Polytope	EJA
$A_n(n \ge 1)$	$SL(n+1,\mathbb{C})/SU(n+1)$	n(n+2)	\triangle_n	Herm $(n+1,\mathbb{C})$
$B_n(n \ge 2)$	$SO(2n+1,\mathbb{C})/SO(2n+1)$	n(2n+1)	\Box_n, \Diamond_n	
$C_n(n \ge 3)$	$Sp(n,\mathbb{C})/Sp(n)$	n(2n+1)	\Box_n, \Diamond_n	
$D_n(n \ge 4)$	$SO(2n,\mathbb{C})/SO(2n)$	n(2n - 1)	⊘n	
F ₄	$F_4^{\mathbb{C}}/F_4$	52	24-cell	
G ₂	$G_2^{(\mathbb{C})}/G_2$	14	hexagon	

Remark:

The non-EJA cases in all three tables, especially the ones occuring in infinite families, should be interesting state spaces to study.

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Inner products, internal representation of the dual and self-duality

Pick any inner product $\langle .,.\rangle$ on A.

- Internal dual of V_+ relative to this inner product is $V_+^{*int} := \{y \in A : \forall x \in V_+ \langle y, x \rangle \ge 0\}$. Of course it is isomorphic to V_+^* (via the iso that takes $y \in A$ to the functional $x \mapsto \langle y, x \rangle$).
- If there is an inner product relative to which V^{*int}₊ = V₊, A is called self-dual.
- Self-duality is stronger than V₊ affinely isomorphic to V^{*}₊! (cf. squit)

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Examples

Classical: *A* is the space of *n*-tuples of real numbers; $u(x) = \sum_{i=1}^{n} x_i$. So Ω_A is the probability simplex, V_+ the positive (i.e.nonnegative) orthant $x : x_i \ge 0, i \in 1, ..., n$

Quantum: $A = \mathscr{B}_h(\mathbf{H})$ = self-adjoint operators on complex (f.d.) Hilbert space \mathbf{H} ; $u_A(X) = \text{Tr}(X)$. Then Ω_A = density operators. V_+ = positive semidefinite operators.

Squit (or P/Rbit): Ω a square, V_+ a four-faced polyhedral cone in \mathbb{R}^3 .

Inner-product representations: $\langle X, Y \rangle = \text{tr } XY$ (Quantum) $\langle x, y \rangle = \sum_i x_i y_i$ (Classical)

Quantum and classical cones are self-dual! Squit cone is not, but is isomorphic to dual.

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Faces of convex sets and Yes/No questions

Face of convex *C*: subset *S* such that if $x \in S \& x = \sum_i \lambda_i y_i$, where $y_i \in C$, $\lambda_i > 0$, $\sum_i \lambda_i = 1$, then $y_i \in S$. **Exposed face**: intersection of *C* with a supporting hyperplane.

- Classical: exposed faces of the orthant correspond to subsets *S* of the indices, and consist of nonnegative *n*-tuples that are zero on all indices not in *S*.
- Quantum: exposed faces of V₊ (resp. Ω) correspond to subspaces of Hilbert space, and consist of the PSD operators (resp. density operators) supported only in S.

For $\alpha \in V_+^*$, ker α is a supporting hyperplane to V_+ . So for any effect e, $e^0 := \{x \in V_+ (\text{resp. } \Omega) : e(x) = 0\}$ is an exposed face of $V_+ (\text{resp. } \Omega)$. So is $e^1 := \{x \in V'_+ : e(x) = 1\}$.

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Consequences of Spectrality and Strong Symmetry

Structure theory of strongly symmetric spectral convex sets from (BMU 2013):

- No-restriction: Even if define "frame" (and hence spectrality and strong symmetry) in terms of ℰ, SSS implies ℰ = [0, u].
- *Self-duality of V*₊. (Mueller and Ududec, PRL: no-restriction + transitivity on 2-frames implies self-duality.)
- Perfection: every face is self-dual in its span according to the restriction of the inner product to that span
- Every face of Ω is generated by a frame. If *F* ≤ *G*, a frame for *F* extends to one for *G*. All frames for *F* have same size, called the rank of *F*.
- The orthogonal (in self-dualizing inner product) projection onto the span of a face *F* is positive, in fact it's a *filter* (defined soon).
- Face lattice is orthomodular.

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The lattice of faces

- Lattice: partially ordered set such that every pair of elements has a least upper bound x ∨ y and a greatest lower bound x ∧ y.
- The faces of any convex set, ordered by set inclusion, form a lattice. So do its exposed faces.
- Face lattices, and exposed face lattices, of V₊ and Ω are isomorphic (up to a trivial convention), via Ω ⊇ F → Cone (F) ⊆ V₊.
- Complemented lattice: bounded lattice in which every element x has a complement: x' such that x ∨ x' = 1, x ∧ x' = 0. (Remark: x' not necessarily unique.)
- orthocomplemented if equipped with an order-reversing complementation: x ≤ y ⇒ x' ≥ y'. (Remark: still not necessarily unique.)
- Orthocomplemented lattices satisfy DeMorgan's laws.

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Orthomodularity

- Orthomodularity: $F \leq G \implies G = F \lor (G \land F')$.
- Interpret as G ∧ F' as a relative orthocomplement (of F relative to G).
- For a face *F* of *V*₊, define the face *F'* as {*y* ∈ *V*₊ : ⟨*y*,*F*⟩ = 0}, (i.e. *F'* = *F*[⊥] ∩ *V*₊). (Transfers to Ω trivially.)
- SSS ⇒ that ' is an orthocomplementation, and the face lattice is orthomodular.

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Further consequences of postulates 1 and 2:

Let e₁,..., e_n be a frame generating *F*. ∑_i e_i =: u_F is *independent* of the frame, and is a projective unit, i.e. P^{*}u_A for *P* a filter (defined in next slide). "Am I in the face *F*?"

• For
$$F = \Omega$$
, $\sum_i e_i = u_A$.

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Filters

Filter: normalized positive linear map $P : V \rightarrow V P^2 = P$, with *P* and P^* both complemented.

Complemented means \exists positive idempotent P' such that $\operatorname{im}_{+}P = \operatorname{ker}_{+}P'$. (States "pass P" iff they "fail P'".) **Normalized** means $\forall \omega \in \Omega \ u(P\omega) \leq 1$.

- Dual of Alfsen and Shultz' (*Geometry of State Spaces of Operator Algebras*, Birkhauser 2003) notion of **compression**.
- Filters are **neutral**: $u(P\omega) = u(\omega) \implies P\omega = \omega$.
- Ω called **projective** if every face is the positive part of the image of a filter.

Quantum example: $\rho \mapsto Q\rho Q$ where Q is the orthogonal projector onto a subspace of Hilbert space \mathcal{H} .

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Flags and regularity

Definition (Farran/Robertson, Madden/Robertson)

- A *flag* Φ of a compact convex set Ω is a sequence *F_i* of nonempty exposed faces such that *F_i* ⊆ *F_{i+1}*.
- It is *maximal* if it is not a proper subsequence of any other flag.
- Ω is *regular* if Aut Ω acts transitively on the set of its maximal flags.

Remark: Farran and Robertson's general definitions and results involve convex bodies in Euclidean spaces and the *symmetry group* of rigid transformations, rather than the in general larger group of affine transformations, but the notions are equivalent for regular convex bodies and we ignore the difference for the purposes of this talk.

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Lemma (Barnum/Hilgert 2019)

Strongly symmetric compact convex sets are regular.

Proof defines, for each maximal frame $\{\omega_1, ..., \omega_r\}$, the sequence of faces $F_1, ..., F_r$

$$F_i := \bigvee_{j=1}^{\prime} \omega_j, \tag{1}$$

shows that this is a maximal flag and that (1) puts maximal frames and maximal flags in bijective correspondence, and that transitive action of Aut Ω on flags implies, via this correspondence, transitive action on frames, using the following elementary fact that does not require SSS:

Proposition

Let g be an automorphism of Ω . If F is a face of Ω , then so is g.F, and g.c(F) = c(g.F). Let Φ be a flag of Ω . Then $g.\Phi$ is a flag; it is maximal if and only if Φ is.

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Fundamental region and Farran-Robertson polytope Definition (Farran-Robertson 1994)

 $\Delta \subseteq \Omega$ is called a **fundamental region** for the action of Aut Ω on Ω if $\Omega = (Aut \ \Omega).\Delta$ and Aut Ω -orbits meet the interior of Δ in at most one point.

Theorem (Farran/Robertson 1984)

Let $\Omega \subseteq E$ be regular and $\Phi = F_1, ..., F_r$ be a maximal flag, and write $c_j := c(F_j)$, the barycenter of F_j . Then

$$\Delta_{\Phi}(\Omega) := \operatorname{Conv} \left\{ c(F_1), c(F_2), ..., c(F_r) \right\}$$
(2)

is a fundamental region. Moreover, $\pi_{\Phi}(\Omega) := \Omega \cap \lim \{c_1, ..., c_r\}$ is a regular polytope, which we call the **Farran-Robertson polytope** of Ω .

Obviously, $\Omega = (Aut \ \Omega).\pi(\Omega)$. Less obviously, for any vertex of $\pi(\Omega)$, we have $\Omega = Conv Aut \ \Omega.\omega$.

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Examples of Farran-Robertson polytopes and fundamental regions

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Proposition (Madden & Robertson 1995; see also Farran & Robertson, Theorem 10, and its proof)

Let Ω be a regular convex body with symmetry group K. Let F be a face of $\pi(\Omega)$ with centroid c(F), and write $K^{c(F)}$ for the isotropy subgroup of K at c(F). Then the orbit $K^{c(F)}$. F is a face of Ω , which we call H_F , and each face of Ω is of the form $k.H_F$ for some face F of $\pi(B)$ and some $k \in K$. Moreover, if $F_1, F_2, ..., F_r$ is a maximal flag of $\pi(\Omega)$, then $H_{F_1}, H_{F_2}, ..., H_{F_r}$ is a maximal flag of Ω , and every maximal flag of Ω arises from a flag of $\pi(\Omega)$ in this way.

Lemma (Barnum/Hilgert 2019)

If Ω is spectral and strongly symmetric, then $\pi(\Omega)$ is a simplex whose vertices form a maximal frame.

Madden and Robertson classified the regular convex bodies and determined $\pi(B)$ for each such body *B*. Our two Lemmas imply that the SSS compact convex sets are the subset of these such that $\pi(B)$ is a simplex. We establish our main theorem by showing that these are precisely the EJA state spaces claimed in that theorem.

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Bits

A system whose largest frame has cardinality 2 we'll call a **bit**. For the regular bits, i.e. those with $\pi(\Omega) = \triangle_1$, we avoid examining cases in the tables by using a general result:

Proposition

Spectral strongly symmetric systems whose maximal frames have cardinality 2 (i.e. SSS "bits") are affinely isomorphic to Euclidean balls.

The proof is essentially due to Dakic and Brukner (2011), who made a similar statement but with the weaker assumptions of spectrality and transitivity of Aut Ω on pure states; their argument made an implicit assumption in one step, which holds if we have transitivity on 2-frames but is not obvious otherwise.

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Madden-Robertson classification of regular convex bodies Theorem

Let G/K be an irreducible noncompact symmetric space, the compact group K connected, K ~ p the isotropy representation, *i.e.* g = t+p is the Cartan decomposition of g := lie G and K ~ p is the restriction to K of the corestriction to p of the adjoint action G ~ g. Let a be a maximal abelian subspace of p and let v ∈ a be such that P := Conv W_K.v ⊂ a is a regular polytope, where W_K := K_a/K^a is the Weyl group of K. Then B := K.P = Conv K.v is a nonpolytopal regular convex body, P = π(B), and P = a ∩ B. B is determined, up to affine isomorphism, by the affine isomorphism class of π(B) and the symmetric space.

Conversely, for every nonpolytopal regular convex body B there is an irreducible noncompact symmetric space G/K such that B = Conv K.v in the isotropy representation K ~ p of compact connected K. π(B) is α∩ B, and is a W_K-orbitope.

Madden-Robertson classification, continued

Theorem (continued)

- In the above, v can be any strictly positive multiple of any fundamental weight w for which Conv W_K.w is a regular polytope. Fundamental weights are specified by marking a node of the Coxeter diagram, and Coxeter indicated which marked nodes give weights such that Conv W_K.w is a regular polytope.
- The list of irreducible noncompact symmetric spaces, with their dimensions, ranks, and the regular polytopes that occur as Weyl orbitopes in Cartan subspaces of their isotropy representations, is given in Tables 2, 3 and 4, which except for their last column are essentially Tables 2,3, and 4 of Madden/Robertson.

Madden-Robertson classification: tables

Table: Classical noncompact symmetric spaces with associated isotropy representations and Farran-Robertson polytopes (adapted from Madden/Robertson, Table 2)

		Rank of symmetric	Dimension of isotropy			
Туре	Symmetric space G/K	space	representation	Root space	Polytope	EJA
AI	$SL(n,\mathbb{R})/SO(n)$	<i>n</i> – 1	(n-1)(n+2)/2	A_{n-1}	\triangle_{n-1}	Herm(n,ℝ)
All	$SU^*(2n)/Sp(n)$	<i>n</i> – 1	(n-1)(2n+1)	A_{n-1}	\triangle_{n-1}	Herm(n,⊞)
AIII	$SU(p,q)/S(U_p \times U_q)$	q	2pq	$\begin{cases} C_q & (q < p) \\ B_q & (q = p) \end{cases}$	\Box_q, \Diamond_q	$\mathbb{R}^2 \oplus \mathbb{R} \ (p = q = 1)$
BI	$SO_0(p,q)/(SO(p) imes SO(q)) \ p+q ext{ odd}, \ q < p$	q	pq	Bq	\Box_q, \Diamond_q	$\mathbb{R}^{p} \oplus \mathbb{R} \ (q=1)$
DI	$SO_0(p,q)/(SO(p) imes SO(q)) \ p+q$ even	q	pq	$\begin{cases} B_q & (q < p) \\ D_q & (q = p) \end{cases}$	\Diamond_q	$\mathbb{R}^{p}\oplus\mathbb{R} \ (q=1)$
DIII	<i>SO</i> *(2 <i>n</i>)/ <i>U</i> (<i>n</i>)	$\lfloor n/2 \rfloor = q$	<i>n</i> (<i>n</i> -1)	$\begin{cases} C_q & q \text{ odd} \\ BC_q & q \text{ even} \end{cases}$	\Box_q, \Diamond_q	$\mathbb{R}^2 \oplus \mathbb{R} \ (q=1)$
CI	$Sp(n,\mathbb{R})/U(n)$	n = q	n(n+1)	C_q	\Box_q, \Diamond_q	$\mathbb{R}^2 \oplus \mathbb{R} \ (q=1)$
CII	Sp(p,q)/(Sp(p) imes Sp(q))	q	4 <i>pq</i>	$\left\{egin{array}{ll} C_q & (q=p) \ BC_q & (q < p) \end{array} ight.$	\Box_q, \Diamond_q	$\mathbb{R}^4 \oplus \mathbb{R} \ (p = q = 1)$

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Table: Exceptional noncompact symmetric spaces with associated isotropy representations and Farran-Robertson polytopes (adapted from Madden/Robertson, Table 3)

Туре	Symmetri	ic space <i>G/K</i> ted by g, ŧ	Rank of symmetric space	Dimension of isotropy representation	Root space	Polytope	EJA
EIII	€6(-14)	so(10)⊕ℝ	2	32	B ₂		
EIV	€6(-26)	f4	2	26	A ₂	∆ ₂	<i>Herm</i> (3, ^ℚ)
EVI	€7(-5)	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	4	64	F ₄	24-cell	
EVII	¢7(-25)	$\mathfrak{e}_6 \oplus \mathbb{R}$	3	54	C_3	□3,◊3	
EIX	€8(-24)	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	4	112	F ₄	24-cell	
FI	f4(4)	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	4	28	F ₄	24-cell	
FII	f4(-20)	so(9)	1	16	A ₁	\triangle_1	
G	\$2(2)	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	2	8	G ₂	hexagon	

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Table: Noncompact symmetric spaces arising as $K^{\mathbb{C}}/K$ for simple K, with associated isotropy representations and Farran-Robertson polytopes (Madden/Robertson, Table 4)

Type and root space	Symmetric space	Dimension of isotropy representation	Polytope	EJA
$A_n(n \ge 1)$	$SL(n+1,\mathbb{C})/SU(n+1)$	n(n+2)	\triangle_n	Herm $(n+1,\mathbb{C})$
$B_n(n \ge 2)$	$SO(2n+1,\mathbb{C})/SO(2n+1)$	n(2n+1)	\Box_n, \Diamond_n	
$C_n(n \ge 3)$	$Sp(n,\mathbb{C})/Sp(n)$	n(2n+1)	\Box_n, \Diamond_n	
$D_n(n \ge 4)$	$SO(2n,\mathbb{C})/SO(2n)$	n(2n - 1)	⊘n	
F ₄	$F_4^{\mathbb{C}}/F_4$	52	24-cell	
G ₂	$G_2^{(\mathbb{C})}/G_2$	14	hexagon	

Remark:

The non-EJA cases in all three tables, especially the ones occuring in infinite families, should be interesting state spaces to study.

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Polar representations and regular convex bodies

The results of Farran/Robertson and Madden/Robertson use the notion of **polar representation** introduced by Jiri Dadok:

Let *G* be a compact connected Lie group. A finite-dimensional real representation $G \frown V$ is called **polar** if it admits a **Cartan subspace:** a linear subspace that meets every orbit orthogonally.

Dadok classified the irreducible polar representations, showing that they are all symmetric space isotropy representations, or subrepresentations of symmetric space isotropy representations having the same orbits.

Proposition (Madden/Robertson (1995), Proposition 2.2 and its proof)

Let B be a regular n-solid in $E \simeq \mathbb{E}^n$, with centroid 0, and let π be the inclusion of the symmetry group G of B in $O(n) \simeq O(E)$. Then the representation π is polar, and for any maximal flag Φ of B, the centroids of the faces in Φ are a basis for a Cartan subspace, L_{Φ} .

From strongly symmetric spectral state spaces to complex quantum mechanics More motivation:

• Adding *energy observability* (BMU '14) or the closely related *orientation* (Connes) or *dynamical correspondence* (Alfsen & Shultz) gives the space of standard quantum density matrices.

Energy observability: Systems have nontrivial continuously parametrized reversible dynamics. Each generator of a one-parameter continuous subgroups ("Hamiltonians") are associated a nontrivial observable conserved by the subgroup. (Recalls Noether's theorem. Relation to a moment map?)

• So does assuming the existence of a *tomographically local* composite $\Omega \otimes \Omega$ that is also strongly symmetric and spectral (SSS). This follows from Masanes and Müller 2011 New J Phys **13** (arxiv:1004.1483)) plus verification that SSS implies their assumptions.

Barnum, Hilgert

Alfsen-Shultz derivation of Jordan algebraic systems

Theorem (Adaptation of Alfsen & Shultz, Thm 9.3.3)

Let a finite-dimensional system satisfy

- (a) there is a filter onto each face
- (b) symmetry of transition probabilities, and
- (c) filters P preserve purity: if ω is a pure state, then P ω is a nonnegative multiple of a pure state.

Then Ω is the state space of a formally real Jordan algebra.

Recall that (a) and (b) follow from SSS.

Alternatives to purity preservation

Alternatives to (c) (Purity preservation) for getting Jordan (all from Alfsen-Shultz):

(c') **Covering law** for face lattice: For every element *F* and atom *a*, either $F \lor a = a$ or $F \lor a$ covers *a*. (An element *b* of lattice **covers** element *a* if $a \le b$ and there is nothing between them. An **atom** is an element that covers 0.

(c'') Every face generated by a pair of pure states is a Euclidean ball.

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Symmetry of transition probabilities

• Given projectivity, for each atomic projective unit $p = P^*u$ (*P* an **atomic** (:= minimal nonzero) filter) the face $P\Omega$ contains a single pure state, call it \hat{p} . $p \mapsto \hat{p}$ is 1:1 from atomic projective units onto extremal points of Ω (pure states).

• Symmetry of transition probabilities says: for all pairs a, b of atomic projective units, $a(\hat{b}) = b(\hat{a})$.

Lemma

A self-dual projective cone has symmetry of transition probabilities.

Proof: With self-dualizing inner product, normalized so $\langle u, \omega \rangle = 1$ for any $\omega \in \Omega$, the map $x \mapsto \hat{x}$ is the identity map on *V*, so $a(\hat{b}) \equiv \langle a, \hat{b} \rangle = \langle a, b \rangle = \langle b, a \rangle = \langle b, \hat{a} \rangle \equiv b(\hat{a})$.

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Energy Observability

Definition

Let A, Ω be a system with a group of reversible transformations \mathscr{G}_A having non-trivial Lie algebra \mathfrak{g}_A .

Energy observable assignment: injective linear map $\varphi : \mathfrak{g}_A \to A^*$ such that

•
$$\varphi(X) \circ X = 0$$
 for all $X \in \mathfrak{g}_A$

•
$$U_A \not\in \operatorname{ran}(\varphi)$$
.

We say that "energy is an observable" in *A* if $g_A \neq \{0\}$ and if there exists an energy observable assignment.

Theorem (BMU 2013)

A finite dimensional EJA state space satisfies energy observability iff it is a standard quantum system (over a complex Hilbert space).

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Local Tomography vs. Energy Observability

• SSS and Local Tomography likely also give standard quantum in a suitable framework.

Question: Relation of LT to energy observability?

• SSS + local tomography + existence of a state space that is a ball give standard quantum.

Thermodynamics and statistical mechanics

Thermo/stat mech phenomena are a natural arena for information-related principles to play a role in physics.

Thermodynamic protocols (e.g. for moving between nonequilibrium states using adiabatic and isothermal processes at cost governed by E - TS in some limit) tend to involve

- spectra, provided by Spectrality,
- use plenty of reversible transformations, provided by Strong Symmetry (but maybe weakenings of SS suffice?),
- possibly measurements using *filters* (Maxwell's demon?), provided by SSS.
- Association of reversible evolution with conserved energy observable, might be relevant, but thermo-like resource protocols might be possible without it?

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Implications of our result for work on thermo/stat mech phenomena in GPT systems

M. Krumm, H. Barnum, M. Müller, Barrett (1608.0446 New J Phys): SSS \implies probabilities of the measurement outcomes in the spectral frame majorize those for any fine-grained measurement. Hence any concave and Schur-concave function of finegrained measurement outcome probabilities is minimized by the spectral measurement, measurement entropy equals preparation entropy, averaging (with any distribution) over Aut Ω increases these entropies.

Chiribella and Scandolo 1506.00380: same conclusions from [causality and] purification, purity preservation under parallel and sequential composition of operations, and strong symmetry. First 4 of these together imply spectrality.

Present work implies these results concern only irreducible Jordan or classical systems.

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Majorization results without SSS: I

Chiribella and Scandolo (1608.0449 / also New J. Phys): same conclusions for *sharp theories with purification*, which don't necessarily have strong symmetry; however Barnum, Lee, Scandolo, and Selby (1704.05106; *Entropy* **19**, p. 253 (2017)) showed that systems in these theories are Jordan-algebraic (though not necessarily simple-or-classical).

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Majorization results without SSS: II

HB, Markus Mueller, Jonathan Barrett, Marius Krumm (1508.03107; EPTCS **415** (=Proc. 12th QPL)):

Definition

A system has **Unique Spectrality** if every state has a decomposition into perfectly distinguishable pure states and all such decompositions use the same probabilities. A convex compact set Ω is **perfect** if $V_+ := \text{Cone}((\Omega))$ is perfect and for each face *F* of V_+ the orthogonal projection onto lin *F* is normalized.

• Perfection and Unique Spectrality imply that for any state, the probabilities of the measurement outcomes in the spectral frame majorize those for any fine-grained measurement.

We still don't know that such systems must be Jordan algebraic. I think I have counterexamples.

Implication for results on query computation

Projectivity of the state space (i.e. filters \leftrightarrow faces) allows reliable information storage, retrieval. One can develop notions of query, and with further assumptions, of phase, phase kickback etc...

• C. Lee and J. Selby, New J. Phys 18 093047 (2016): with assumptions of causality, purification (unique up to reversible operations on the purifying systems), purity preservation under composition (apparently only parallel composition is used), the existence of a pure sharp effect, and strong symmetry, obtaining success probability > 1/2 in Grover's "search" problem requires at least $(3/2 - \sqrt{2})\sqrt{N/h}$ gueries, where h is the maximal degree of "higher order interference" possible in the systems used. If h is independent of the problem size N, this is $\Omega(\sqrt{N})$, no better than the quantum bound. Since the first four assumptions imply spectrality, our result shows this is actually a Jordan algebraic setting, where h can be at most 2 (Niestegge 2012; Barnum and Ududec (unpublished) 2009).

Query complexity, II

Barnum, Lee, and Selby (*Found. Phys.* **48**, 954-981 (2018); arxiv:1704.05043) obtained two results on GPT query complexity in a similar setting

(1): a generalization of a quantum "zero-information" query lower bound of Meyer and Pommersheim: if the maximal order of interference is *h*, then if *hn* quantum queries yield no information about which of several classes a queryable function falls into ("classification problem") then neither do *n* GPT queries, and (2) confirmation that the notion of black-box query used is reasonable: if there is a polynomial-size family of GPT circuits C_f for a family of functions *f*, one can use them to simulate the black-box queries to the functions *f*, with a polynomial family of circuits.

Again, the present result shows that these results are confined to the setting of irreducible Jordan-algebraic, and classical, systems.

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