Spectral Properties of Convex Bodies

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Abstract. We use the Madden-Robertson classification of regular convex bodies to show that convex bodies are *spectral* and *strongly symmetric* if and only if they are affinely isomorphic to the normalized state spaces of simple euclidean Jordan algebras, or to simplices. Further, we discuss the relevance of this result for *general probabilistic theories* of quantum and classical physical systems, and its relation to other characterizations of various classes of euclidean Jordan algebra state spaces.
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1. Introduction

Quantum mechanics is modelled in terms of Hilbert spaces. Lines appear as pure states and self-adjoint operators as observables. In traditional courses on the subject one deals with infinite dimensional Hilbert spaces and evolution equations such as the Schrödinger equation. In the context of quantum information finite dimensional Hilbert spaces also play an important role. In this paper we restrict our attention to the finite dimensional situation.

Fix a finite dimensional complex Hilbert space $\mathcal{H}$. A *state* is a *density operator* on $\mathcal{H}$, i.e. a self-adjoint operator $\rho \in \mathrm{End}(\mathcal{H})$ such that $\rho \geq 0$ and $\mathrm{Tr}(\rho) = 1$ (trace). The set of states is compact and convex. Its extremal points are then called the *pure states*. As operators they are orthogonal projectors to complex lines in $\mathcal{H}$. Note here that the self-adjoint operators in $\mathrm{End}(\mathcal{H})$ form a Jordan algebra with respect to the product $a \cdot b := \frac{1}{2}(ab + ba)$. The space of self-adjoint operators is viewed as its own dual space via the trace inner product. Self-adjoint operators that are positive semidefinite and below the identity operator in the ordering induced by the positive semidefinite cone are known as *effects*: viewing them as affine functionals evaluated on states, they give probabilities. The orthogonal projection operators are effects; indeed they are the extremal points of the convex set of effects. They also appear as the elements into which any self-adjoint operator has a spectral expansion.

Thus the probabilistic interpretation of quantum mechanics is closely related to the spectral theory of density matrices. A research field at the intersection of physics and mathematics studies what have come to be called *General Probabilistic*

*Dedicated to Jimmie Lawson

Theories (GPTs), in which compact convex sets are used to model the state spaces of abstract (not necessarily quantum) physical or other systems, and affine functionals on the affine span of the convex state space give probabilities of measurement outcomes or other processes when evaluated on states. It is therefore deeply concerned with properties of compact convex sets—including the corresponding spectral theory when such a theory is possible—abstracted from the above example. Although strictly speaking the main result of the present paper is purely a proposition in the convex geometry of finite-dimensional compact convex sets, it is partly motivated by the GPT framework and is formulated using concepts arising in that framework. It says that a natural spectral assumption together with the presence of a strong symmetry condition automatically imply the presence of a Jordan algebra structure.

The main result of the paper is presented and proved in Sections 2-6. Using a basic mathematical framework presented in the remainder of this introduction, Sections 2 and 3 develop the specific mathematical apparatus and concepts needed for the main theorem, whose formal statement is then given at the end of Section 3. Section 4 describes important results, mostly from [13], concerning the structure of state spaces of strongly symmetric spectral convex sets, that are crucial to proving our main theorem. In Section 5, we prove two crucial lemmas on the structure of compact convex sets as regular compact convex sets, and in Section 6 we prove the main theorem by using these lemmas to place strongly symmetric spectral sets within the Madden-Robertson classification [49] of regular compact convex sets.

The last three sections discuss implications of our result and relate it to other work. Section 7 situates our work in the context of general probabilistic theories, and describes implications for some existing results that use the assumptions of strong symmetry and spectrality. Section 8 relates it to other characterizations of various classes of Jordan-algebraic state spaces, and Section 9 describes various known properties which can serve to narrow down various classes of Jordan-algebraic state spaces to standard quantum state spaces, i.e. the spaces of complex hermitian $n \times n$ density matrices, or direct sums of such spaces.

In the remainder of this introduction we describe the basic mathematical setting for our main result.

Let A be a finite dimensional affine space over $\mathbb{R}$. We view A as an affine hyperplane of the form $u^{-1}(1)$ in a real vector space V, where $u \in V^*$ is a linear functional on V. We fix a compact convex subset $\Omega \subseteq A$ which spans A as an affine space. By $V_+ := \mathrm{Cone}(\Omega) \subseteq V$ we denote the (automatically convex) cone spanned by Ω in V. Note that $u(\Omega) = \{1\}$ implies that $u \in V_+^*$, where $V_+^* := \{x \in V^* \mid \forall v \in V_+ : x(v) \geq 0\}$ is the *dual cone* of V_+. Note that V_+^* defines a partial order on V^*, which we will denote by $\leq$. Note also that the structure is determined completely by the triple (V, Ω, u).

Definition 1.1. We call the triples (V, Ω, u) *embedded state spaces.* An embedded state space is *irreducible* if it cannot be written in the form $(V_1 \oplus V_2, \Omega_1 \oplus \Omega_2, u_1 + u_2)$ for two embedded state space triples (V_1, Ω_1, u_1) and (V_2, Ω_2, u_2).

Although we make extensive use of the embedding of Ω in V and of the cone V_+, our result concerns only the convex geometry of Ω and hence is independent of

the particular choice of embedding. This is because of the following elementary fact:

Proposition 1.2. *Let (V, Ω, u) and (W, Υ, v) be two embedded state spaces. Ω is affinely isomorphic to Υ if, and only if, there is an isomorphism $\phi : V \to W$ of ordered linear spaces such that $\phi(\Omega) = \Upsilon$.*

We note in passing that since in the triple (V, Ω, u), Ω spans an affine hyperplane in $V \setminus \{0\}$, the data (V, Ω) determine u.

2. Distinguishability and frames

In this section we formulate the conditions on state spaces that, according to our main result, ensure the presence of Jordan structure.

We consider the group $\mathrm{Aut}(A)$ of affine automorphisms of A as a subgroup of GL(V) and set

$$K := \mathrm{Aut}(\Omega) := \{\varphi \in \mathrm{Aff}(A) \mid \varphi(\Omega) = \Omega\}.$$

Remark 2.1. The space A can be given the structure of a euclidean vector space E such that $0 \in E$ is the barycenter $c(\Omega)$ of Ω and $\mathrm{Aut}(\Omega) \subseteq \mathrm{O}(E)$. Of course V can also be given a euclidean structure such that $\mathrm{Aut}(\Omega)$, considered as a subgroup of $\mathrm{GL}(V)$, is a subgroup of $\mathrm{O}(V)$ fixing the one-dimensional subspace $\mathbb{R}c(\Omega)$.

Definition 2.2. (i) An element $x \in V^*$ is called an *effect* if it is contained in the *order interval* $[0, u] := \{x \in V^* \mid 0 \leq x \leq u\}$.

(ii) A *measurement* is a finite sequence $e_1, \ldots, e_m$ of effects such that $\sum_{j=1}^m e_j = u$.

(iii) The elements of Ω are called *states*. The extremal states are called *pure*.

(iv) The states $\omega_1, \ldots, \omega_m \in \Omega$ are called *perfectly distinguishable* if there exists a measurement $e_1, \ldots, e_m$ such that $e_i(\omega_j) = \delta_{ij}$ for $i, j \in \{1, \ldots, m\}$.

(v) An m-tuple $(\omega_1, \ldots, \omega_m)$ of pure states is called a *frame* (or m-frame) if the ω_j are perfectly distinguishable.

An equivalent characterization of perfect distinguishability will be used later. We call an indexed set of effects $E = \{e_i\}$ a *submeasurement* if $\sum_i e_i \leq u$. For each submeasurement $E = \{e_1, \ldots, e_r\}$, the indexed set $E' := \{e_1, \ldots, e_r, e_{r+1}\}$, where $e_{r+1} := u - \sum_{i=1}^r e_i$, is a measurement. Then perfect distinguishability of $\omega_1, \ldots, \omega_r$ is equivalent to the existence of a submeasurement such that $e_i(\omega_j) = \delta_{ij}$, for it is easy to see that $e_{r+1}(\omega_i) = 0$ for all $i \in \{1, \ldots, r\}$, whence there are many ways of constructing an r-outcome measurement $\{e'_1, \ldots, e'_r\}$ such that $e_i(\omega_j) = \delta_{ij}$—for instance, let $e'_1 = e_1 + e_{r+1}$, and $e'_i = e_i$ for all $i \in \{2, \ldots, r\}$.

Now we are able to formulate the two conditions on Ω which, by our main result, will enforce its affine isomorphism to the normalized state space of a Jordan algebra, and hence the existence of an embedding (V, Ω, u) such that V has the structure of a Jordan algebra with unit $e = u$:

Definition 2.3. (i) Ω is called *spectral* if for each $\omega \in \Omega$ one can find a frame $(\omega_1, \dots, \omega_m)$ such that ω is contained in the convex hull $\mathrm{Conv}(\omega_1, \dots, \omega_m)$.

(ii) Ω is *strongly symmetric* if for all $m = 1, \dots, 1 + \dim \Omega = \dim V$ the natural action of $\mathrm{Aut}(\Omega)$ on the set of m-frames is transitive.

Note that the transitive action of $\mathrm{Aut}(\Omega)$ on frames includes the ability to permute the elements of a given frame. Also note that strong symmetry implies *pure-state transitivity*, i.e. transitive action of $\mathrm{Aut}(\Omega)$ on the set $\partial_e\Omega$ of extreme points of Ω, and the latter implies that for any $\omega \in \partial_e\Omega$, $c(\Omega) = \int_{\mathrm{Aut}(\Omega)} d\mu(g) g.\omega$, where $g \in \mathrm{Aut}(\Omega)$ and $d\mu$ is normalized Haar measure on $\mathrm{Aut}(\Omega)$.

The following two examples are prototypical for the situation described above.

Example 2.4 (Simplices in $\mathbb{R}^{n+1}$)**.** Let $V = V^* = \mathbb{R}^{n+1}$ with standard inner product and standard basis $e_1, \dots, e_{n+1}$. We set $\omega_j := e_j$ and let A be the affine span of $\omega_1, \dots, \omega_{n+1}$. Then we define $u \in V^*$ by

$$u\Big(\sum_{j=1}^{n+1} c_j e_j\Big) := \sum_{j=1}^{n+1} c_j.$$

For $\Omega := \mathrm{Conv}(\omega_1, \dots, \omega_{n+1})$ the ω_j are the pure states and $e_1, \dots, e_{n+1}$ is a measurement. Moreover $\mathrm{Aut}(\Omega)$ is simply the symmetric group $\mathcal{S}_{n+1}$ permuting the pure states. Thus $(\omega_1, \dots, \omega_{n+1})$ are an $n+1$-frame and all other $n+1$-frames are permutations of this one. As $\mathcal{S}_{n+1}$ acts transitively on subsets of fixed cardinality we see that Ω is spectral and strongly symmetric. In the study of general probabilistic theories, simplices are often called *classical state spaces*.

Example 2.5 (Real symmetric matrices)**.** Let V be the space of real symmetric $n \times n$-matrices and $V_+ := \{a \in V \mid a \text{ positive semidefinite}\}$. Set $u := \mathrm{tr} := \frac{\mathrm{Tr}}{n}$ and

$$\Omega := \{a \in V_+ \mid a \text{ positive semidefinite}, \mathrm{tr}(a) = 1\}.$$

If we identify V and V^* via the trace form we obtain $u = \mathbf{1}_n$, the identity matrix of size n. Then measurements are decompositions of $\mathbf{1}_n$ into sums of positive semidefinite matrices. In particular, any decomposition of $\mathbf{1}_n$ into a sum of orthogonal projections is a measurement. An n-frame is a decomposition of $\mathbf{1}_n$ into sums of orthogonal rank-1 projections.[1] In this case $\mathrm{Aut}(\Omega) = \mathrm{O}(n)/\{\pm\mathbf{1}_n\}$ with respect to the action $a \mapsto gag^\top$ for $g \in \mathrm{O}(n)$. Now standard linear algebra implies that Ω is spectral and strongly symmetric.

For readers interested in seeing an example that is nonclassical, but quite different in nature from the quantum and Jordan-algebraic state spaces that are our main concern here, we give the following example which is, however, not essential to understanding our main result.

[1] This is not necessarily obvious, but it follows from Lemma 3.3 below.

Example 2.6. The *square bit*, sometimes called *squit* or *gbit* (the latter for "generalized bit") is the system whose normalized state space Ω is a square. If one represents this as the convex hull of $(0,0),(0,1),(1,0),(1,1)$ in $\mathbb{R}^2$, one may interpret the coordinates (x,y) as two probabilities: x is the probability of getting the outcome x_1 in a measurement with two outcomes x_1, x_2, and similarly y is the probability of getting the outcome y_1, in a second possible two-outcome measurement that can be made on the system. The representation (V,Ω,u) is of course not unique, but one may take $V=\mathbb{R}^3$, let Ω be the convex hull of $(-\frac{1}{2},-\frac{1}{2},1),(-\frac{1}{2},\frac{1}{2},1),(\frac{1}{2},-\frac{1}{2},1)$, and $(\frac{1}{2},\frac{1}{2},1)$, with $u=(0,0,1)$. The cone V_+ over Ω is polyhedral, with four extremal rays and four maximal proper faces. If we use the dot product to represent linear functionals as elements of $\mathbb{R}^3$, the convex body of effects $\mathcal{E}$ is the convex hull of $(0,0,0),(0,0,1)$, and the four additional extremal effects $a=(1,0,\frac{1}{2}), b=(0,1,\frac{1}{2}), c=(-1,0,\frac{1}{2}), d=(0,-1,\frac{1}{2})$. It is shaped like two pyramids, with apices $(0,0,1)$ and $(0,0,0)$, glued together at their square bases whose corners are the four effects just listed, and generates the dual cone, which is also a polyhedral cone with four extremal rays, but rotated by $\pi/4$ (around the z-axis) from the cone over Ω. The measurement x mentioned above has outcome probability for x_1 given by a for x_2 given by c: similarly measurement y corresponds to effects b and d. This state space has been important in the study of generalized probabilistic theories, as an example of how a system may have some, but not all, of the properties sometimes considered to be peculiarly quantum. In particular, while the measurements x and y are complementary in the sense that in general a measurement of one must disturb the outcome probabilities for the other, all of the pure states guarantee definite outcomes—no uncertainty—for *both* of these measurements.

The following result will simplify the proof of our main theorem. This result is essentially proved in [29], where, however, a stronger claim, with the assumption of transitivity on pure states in the premise, is made. A small gap in the proof needs to be filled by the stronger assumption of transitivity on 2-frames.

Proposition 2.7. *If Ω is spectral and strongly symmetric, and such that maximal frames have cardinality less than 3, then Ω is affinely isomorphic to a euclidean ball.*

Proof. Because it is an orbit of a subgroup of $\mathrm{O}(E)$, the set $\partial_e\Omega$ of extremal points of Ω is contained in the sphere $S := \{x \in E : ||x|| = c\}$ where $||\cdot||$ denotes the euclidean norm. We scale the inner product by a positive real number so that $c=1$.

We now prove that every pair of perfectly distinguishable points in Ω are the endpoints of some diameter of S. We begin by showing (following [29] but with a bit more detail) that for every extremal $\omega \in \Omega$, the point $-\omega$ also belongs to Ω, and $[\omega,-\omega]$ is a 2-frame.

Recall that a *chord* of a sphere is defined to be a closed line segment whose endpoints are two distinct points on the sphere. We will use the fact that the only chords of a sphere that contain its center are the diameters, i.e. the chords from x to $-x$. Since Ω is spectral with maximal frame size 2, every nonextremal point in Ω, in particular its center, 0, is a convex combination of two perfectly distinguishable

extremal points of Ω. Let ω_0 and ω_1 be extremal points of Ω such that 0 is a convex combination of them. Since all extremal points of Ω lie on the sphere S, the set of convex combinations of ω_0 and ω_1 is a chord of S containing its center, 0. Therefore it is a diameter, and $\omega_1 = -\omega_0$. Since Ω has pure-state transitivity (i.e. transitivity of $\mathrm{Aut}(\Omega)$ on 1-frames, which are precisely the extremal points), every extremal point ω of Ω can be obtained from ω_0 by acting with an element of $\mathrm{O}(V)$, whence, by linearity of the action, $-\omega$ is also in Ω. So we have established that Ω is symmetric under coordinate inversion $x \mapsto -x$, and that every pair $\omega, -\omega$ is a maximal frame.

We still need to show that there are no other maximal frames in Ω, i.e. no 2-frames that are not the endpoints of a diameter.[2] If we have transitivity on 2-frames, we get this immediately: every 2-frame is an automorphic image of $(\omega_0, -\omega_0)$, and therefore of the form $(\omega, -\omega)$ for some extremal ω.

We follow [29] in using this to show that Ω is a ball. The barycenter (also known as *centroid*) of a compact convex set, which is 0 in the case of Ω, is in its relative interior. Ω is full-dimensional, so its relative interior is its interior, and there is an open ball around 0 contained in Ω. So for any $x \in S$ there is $\lambda \in (0, 1]$ small enough that $\lambda x \in \Omega$. By spectrality, λx is a convex combination of two perfectly distinguishable extremal points of Ω. Since all such pairs are endpoints of diameters, λx must be a convex combination of the endpoints of a diameter. For $x \in S$ the only diameter containing $\lambda x \neq 0$ is the one between x and $-x$. So we have shown that $x \in \Omega$; but x was an arbitrary element of S. Since the entire sphere S belongs to the extreme boundary of the convex set Ω, and we earlier showed that all extremal points of Ω are in S, Ω is the convex hull of the $(n-1)$-sphere $S \simeq S_{n-1}$, i.e. it is an n-dimensional euclidean ball. ■

3. Euclidean Jordan Algebras

Recall the notion of a *euclidean Jordan algebra* (EJA) from e.g. [32]. In a euclidean Jordan algebra V one has the cone V_+ of squares, the intrinsic *Jordan trace* $\mathrm{tr}(x)$ of an element $x \in V$, and inner product $\langle a, b\rangle := \mathrm{tr}(a \cdot b)$, which we use to identify V with V^*. With this identification, $V_+ = V_+^*$, i.e. V_+ is *self-dual* with respect to this inner product. Then $\Omega := \{x \in V_+ \mid \mathrm{tr}(x) = 1\}$ is a compact convex set which is called the *normalized state space* of V, with $\mathrm{Aut}(\Omega) < \mathrm{O}(V)$. In this example the order unit u is equal to the unit e of the Jordan algebra.

Jordan, von Neumann and Wigner [42] classified the finite-dimensional euclidean Jordan algebras. They are precisely the $n \times n$ self-adjoint matrices with entries in $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ and the 3×3 octonionic self-adjoint matrices, equipped in each case with symmetrized matrix multiplication $x \cdot y = (xy + yx)/2$ as Jordan product, and the *spin factors* $\mathbb{R}^n \oplus \mathbb{R}$ for every $n \geq 1$, equipped with the product

$$(\mathbf{x}, s) \cdot (\mathbf{y}, t) = (t\mathbf{x} + s\mathbf{y}, \langle \mathbf{x}, \mathbf{y}\rangle + st). \tag{1}$$

Here $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $s, t \in \mathbb{R}$. Self-adjoint ("hermitian" is also used) means $M = M^\dagger$, where $M^\dagger := \overline{M}^t$, and $\overline{M}$'s entries are the conjugates of M's with respect to the

[2] This is the main point at which we perceive a gap in the argument in [29] which we do not see how to easily bridge using only pure-state transitivity and spectrality.

Table 1: euclidean Jordan algebras with associated cones and Lie algebras

V	V_+	$\mathfrak{g}$	$\mathfrak{k}$	$\dim V$	$\operatorname{rank} V$
$M_m(\mathbb{R})_{sa}$	$PSD(m,\mathbb{R})$	$\mathfrak{sl}(m,\mathbb{R})\oplus\mathbb{R}$	$\mathfrak{o}(m)$	$m(m+1)/2$	m
$M_m(\mathbb{C})_{sa}$	$PSD(m,\mathbb{C})$	$\mathfrak{sl}(m,\mathbb{C})\oplus\mathbb{R}$	$\mathfrak{su}(m)$	m^2	m
$M_m(\mathbb{H})_{sa}$	$PSD(m,\mathbb{H})$	$\mathfrak{sl}(m,\mathbb{H})\oplus\mathbb{R}$	$\mathfrak{su}(m,\mathbb{H})$	$m(2m-1)$	m
$\mathbb{R}\oplus\mathbb{R}^{n-1}$	$Lorentz(1,n-1)$	$\mathfrak{o}(1,n-1)$	$\mathfrak{o}(n)$	n	2
$M_3(\mathbb{O})_{sa}$	$PSD(3,\mathbb{O})$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_4$	27	3

canonical conjugation on $\mathbb{R},\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. The conjugation is the identity in the case of $\mathbb{R}$, thus the self-adjoint real matrices are just the real symmetric matrices. The following table, essentially from [32], gives further information about these Jordan algebras: $\mathfrak{g}$ is the Lie algebra of $\mathrm{Aut}(V_+)$ and the maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, is the Lie algebra of $\mathrm{Aut}(\Omega)$.

A *Jordan frame* for V consists of a maximal orthogonal set of primitive idempotents in V. Jordan frames appear in the spectral theorem for finite-dimensional EJAs (cf. [32]):

Theorem 3.1 (Spectral theorem for finite-dimensional euclidean Jordan algebras)**.** *Every element x of a euclidean Jordan algebra has a decomposition*

$$x=\sum_{i=1}^{r}\lambda_i c_i \tag{2}$$

where $\lambda_i\in\mathbb{R}$ and c_i are a Jordan frame. When we rewrite this as

$$x=\sum_{\alpha}\lambda_\alpha c_\alpha, \tag{3}$$

where $c_\alpha := (\sum_{i\in\alpha} c_i)$ and the sets $\alpha\subseteq\{1,\dots,r\}$ are a partition of the indices into the largest subsets within which $\lambda_i =: \lambda_\alpha$ is constant, then the decomposition (3), into not-necessarily-primitive idempotents, is unique.

This is a combination of Theorems III.1.1 and III.1.3 in [32]. The values λ_α are called the *spectrum* of x.

Proposition 3.2. *The normalized state space of a euclidean Jordan algebra is spectral.*

Proof. The primitive idempotents of an EJA V are precisely the extremal points of its normalized state space Ω (cf. [32]). Since V_+ is self-dual, and all idempotents are below or equal to the order unit e,[3] they are also effects. Orthogonality of primitive idempotents with respect to the inner product implies, given our identification

[3]An easy argument from the spectral theorem gives that every primitive idempotent is part of a Jordan frame, and it is part of the definition of Jordan frame that it sums to the order unit.

of V with V^*, that any subset of a Jordan frame, considered as a set of states, is perfectly distinguished by that Jordan frame, considered as a measurement. Consequently, an ordered subset of a Jordan frame (and in particular, an ordered Jordan frame itself) is a frame in the sense of Definition 2.2 (v). So the spectral theorem for EJAs implies their spectrality. ■

Lemma 3.3. *All frames of the normalized state space of a euclidean Jordan algebra V are tuples consisting of elements of a Jordan frame for V.*

This is known, and implicitly assumed in [13], but we give a proof.

Proof. Since $\partial_e\Omega$ is the set of primitive idempotents, and V_+ is self-dual with respect to the inner product $\langle a, b\rangle = \mathrm{tr}(a \cdot b)$, a frame is a sequence c_i, $i \in \{1, \ldots, s\}$ of primitive idempotents such that there exists a submeasurement e_i for which $\langle e_i, c_j\rangle = \delta_{ij}$. In a general setting, not only in Jordan state spaces, it follows immediately from the condition $\langle e_i, \omega_j\rangle = \delta_{ij}$ on a frame that if $e_i = \sum_k p_k f_k$ is a convex decomposition of e_i into effects f_k, each of the f_k also has the property $\langle f_k, \omega_j\rangle = \delta_{kj}$. So the condition that ω_i is a frame may be restated as the existence of a submeasurement consisting of *extremal* (in the convex body $[0, e]$) effects. Proposition 1.40 of [3] states that the extreme points of the positive part $[0, e]$ of the unit ball of a JB-algebra are the idempotents. The finite-dimensional JB-algebras are the EJAs. It is also known (cf. [3], Proposition 2.18) that for idempotents p_i, $\sum_{i=1}^k p_i \leq e$ implies that $p_i \perp p_j$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$. So in the condition for c_i to be a frame, we may take the e_i to be a mutually orthogonal set of idempotents. We show that $\langle e_i, c_i\rangle = 1$ for an idempotent e_i and a primitive idempotent c_i implies that $c_i \leq e_i$. To do so we use the fact, from [3], that JB-algebras V are equipped with normalized self-adjoint idempotent positive linear maps $P_p : V \to V$ called *compressions*, in bijection with the idempotents p, such that $p = P_p e$ and the exposed faces (all faces in finite dimension) are the positive parts of the images of compressions. We have $1 = \langle e_i, c_i\rangle = \langle P_{e_i} e, c_i\rangle = \langle e, P_{e_i} c_i\rangle$. Since it follows from Proposition 1.41 of [3] (in finite dimensions, where the dual space may be identified with the primal space) that compressions on an EJA are *neutral*, i.e. $||P\omega|| = ||\omega||$ implies $P\omega = \omega$, we have that $P_{e_i} c_i = c_i$, hence $c_i \in \mathrm{im}_+ P_{e_i}$, where the latter is defined as $\mathrm{im}\, P_{e_i} \cap V_+$. By Lemma 1.39 of [3], $\mathrm{im}_+ P_{e_i} \cap [0, e] = [0, e_i]$, and since $c_i \in [0, e]$, we have $c_i \leq e_i$.

With $c_i \leq e_i$, and $e_i \perp e_j$ for all $i \neq j$, it follows that $c_i \perp c_j$ for all $i \neq j$, and consequently that the c_i are a subsequence of an ordered Jordan frame. ■

Now the transitivity properties of Jordan frames allow us to prove strong symmetry.

Proposition 3.4. *The normalized state space of a simple euclidean Jordan algebra is strongly symmetric.*

Proof. Corollary IV.2.7 of Theorem IV.2.5 in [32] states that the compact group K, defined as the subgroup of $\mathrm{Aut}_0(V_+)$ that fixes the Jordan unit e, acts transitively on the set of Jordan frames. Since we've adopted a canonical inner product, this

is a subgroup of $\mathrm{Aut}(\Omega)$. It is clear from the proof of Theorem IV.2.5 in [32] that this transitive action is on *ordered* Jordan frames. Since we showed, in the proof of Lemma 3.2 and in Lemma 3.3, that the frames (in the sense of Definition 2.2) are precisely the ordered subsets of Jordan frames, the group K, and hence $\mathrm{Aut}(\Omega)$, acts transitively on the set of k-frames for each k. ■

Finally, we are ready to formulate our main result.

Theorem 3.5. *If Ω is spectral and strongly symmetric, then either Ω is a simplex or it is affinely isomorphic to the normalized state space of a simple euclidean Jordan algebra.*

That the normalized state space of a simple euclidean Jordan algebra is spectral and strongly symmetric was shown in Propositions 3.2 and 3.4. The proof of the converse requires some more preparation and will be given in Section 6.

4. Structure of strongly symmetric spectral sets

In [13], the class of state spaces that appears in our main theorem was characterized using, in addition to spectrality and strong symmetry, a third postulate: the nonexistence of "higher-order" interference, which, roughly speaking, is probabilistic interference involving three or more mutually exclusive alternatives that is not explainable in terms of pairwise interference between them. Our main result thus improves on [13] by showing that this third postulate was superfluous. However, in [13] many important structural features of Ω and V_+ were shown to follow just from the first two postulates, and we will make crucial use of these results, which we now describe. They use a few additional notions which we first define.

Since we are in finite dimension, a positive definite inner product $\langle\,\cdot\,,\,\cdot\,\rangle : V \times V \to \mathbb{R}$ induces (as does any nondegenerate bilinear form, positive definite or not) an isomorphism between V^* and V. If we use this to identify V with V^* then the dual cone becomes $V_+^* := \{y \in V \mid \forall x \in V_+ : \langle y, x\rangle \geq 0\}$. We say a cone is *self-dual with respect to a given inner product* if $V_+ = V_+^*$ upon making this identification, and we say that a cone V_+ is *self-dual* if there exists an inner product with respect to which it is self-dual.[4] We call such an inner product *self-dualizing*. A property stronger than self-duality is *perfection* (the term is from [5]): a cone V_+ is called *perfect* if there exists an inner product such that every face F of V_+ (including V_+ itself) is self-dual with respect to the restriction of that inner product to F's span.[5] Recall that a face of a convex set is *exposed* if it is obtained from the set by intersecting it with an affine hyperplane.

Definition 4.1. A *complementation* is an involutive map $F \mapsto F'$ of a bounded lattice such that $F \vee F' = 1$ and $F \wedge F' = 0$. It is called an *orthocomplementation*

[4]Self-duality is a properly stronger property than affine isomorphism of V_+^* with V_+, which is sometimes called "weak self-duality". The cone with square base, for example, separates the two properties.

[5]This is properly stronger than every face of a cone being self-dual: the cone with pentagonal base has every face self-dual, but is not perfect.

if it is order-reversing: $F \leq G \Leftrightarrow G' \leq F'$.

Proposition 4.2 (Mostly from [13]). *For a convex compact set Ω that is spectral and strongly symmetric, the following hold:*

1. *Every face of Ω is generated (as a face) by a frame. Any two frames that generate the same face F have the same cardinality, which we call the* rank, *$|F|$, of the face. If the face G is a proper subset of F, then $|G| < |F|$.*

2. *Every face of Ω is exposed.*

3. *The cone $V_+ := \mathbb{R}_+\Omega$ over Ω is a perfect self-dual cone. The self-dualizing inner product $\langle .,. \rangle$ can be chosen to be $\mathrm{Aut}(\Omega)$-invariant, and such that $\langle \omega, \omega \rangle = 1$ for all pure states (extremal points) ω of Ω.*

4. *With respect to the self-dualizing inner product on V, the elements of any frame form an orthonormal set. The states of a frame, viewed as elements of the dual space via this inner product, are effects, and are therefore a distinguishing submeasurement for that frame. If $\omega_1, \ldots, \omega_n$ is a maximal frame, i.e. a frame for Ω, then $\sum_{i=1}^n \omega_i$ is the order unit.*

5. *The face $F = \omega_1 \vee \cdots \vee \omega_k$ generated by a frame $\omega_1, \ldots, \omega_k$ has barycenter $\sum_{i=1}^k \omega_i/k$.*

6. *If F is a face of V_+, (resp. Ω) then $F' := F^\perp \cap V_+$ (resp. $F^\perp \cap \Omega$) is a face of V_+ (resp. Ω) such that $F \wedge F' = \{0\}$ (resp. $F \wedge F' = \emptyset$) and $F \vee F' = V_+$ (resp. $F \vee F' = \Omega$). (Here all $\perp$'s are taken in V, with respect to the self-dualizing, invariant inner product.) In a lattice, these two conditions define what it means for an element F' to be a* complement *of F, so we call F' the face complementary to F, or simply F's complement.*

7. *The face generated by a maximal frame is Ω itself. Every frame A of Ω, generating a face F, extends to a maximal frame M, by appending a frame B for F'. Similarly if $F < G$ (i.e. $F \subsetneq G$), every frame for F extends to a frame for G.*

8. *The map $F \mapsto F'$ on the face lattice of V_+ (equivalently of Ω) is an orthocomplementation, with respect to which the lattice is* orthomodular, *i.e. for $F \leq G$, $G = F \vee (F' \wedge G)$. The additional states appended to a frame on F in order to extend it to a frame on G (cf. item 7 above), are a frame for $F' \wedge G$.*

We may think of orthomodularity as stating that $F' \wedge G$ behaves as a "relative orthocomplement" of F in G, i.e. an orthocomplement in the sublattice consisting of elements below or equal to G.

Proof. Item 1 is Proposition 2 of [13]. Item 2 follows easily from item 1.

Item 3 is established in [13] in the course of proving Theorem 8 of that paper, which states that for every face F, the orthogonal projection (with respect to the self-dualizing $\mathrm{Aut}(\Omega)$-invariant inner product) P_F onto the linear span of a face F, is

a positive map. Iochum [41] showed (see also [5]) that for self-dual cones, positivity of all such projections with respect to a self-dualizing inner product is equivalent to perfection. The self-duality of the cones $\mathbb{R}_+\Omega \subset V$ over strongly symmetric spectral convex sets, with respect to an inner product with the stated properties, was established as Proposition 3 of [13].[6]

Item 4 is Proposition 6 of [13].

Item 6 is partly stated in Proposition 7 of [13], and the rest can be extracted from the proof of that proposition.

Item 7 is part of Proposition 7 of [13].

Item 8. The involutiveness of $'$ is also part of Proposition 7 of [13]. The other two conditions on a complementation are part of item 6 above. The last part of orthocomplementation, order-reversingness, is shown as part of the proof of Theorem 9 in [13]; it follows directly from the extendibility of frames to maximal frames.

The second part of item 8, i.e. orthomodularity, is Theorem 9 of [13]. The crucial element in its proof (given that we have already established that $'$ is an orthocomplement) is the "relative frame extension property", i.e. the last sentence in item 7. The last sentence of item 8 is a step in establishing this relative frame extension property in [13].

The only item we have not covered yet is item 5. It does not appear to be explicitly stated in [13], although it is likely known. It is proved in the course of proving Proposition 4.3 below. ■

Proposition 4.3. *Let Ω be a strongly symmetric spectral compact convex set. Then every face of Ω is a strongly symmetric spectral compact convex set; moreover if F is a face of Ω and $K = \mathrm{Aut}(\Omega)$, then $K(F) := K_F/K^F = \mathrm{Aut}(F)$. Here K_F is the subgroup that takes F to itself; K^F is the subgroup that fixes F pointwise. Also, $K_F = K^{c(F)}$, where $c(F) = \sum_{i=1}^{|F|} \omega_i/|F|$, for any frame ω_i for F, is the centroid of F.*

Proof. From the facts that $c(\Omega)$ is the Haar average over $\partial_e\Omega$, that $u = \sum_{i=1}^r \omega_i$ in a strongly symmetric compact convex set of rank r, where ω_i are a maximal frame, and that u is $\mathrm{Aut}(\Omega)$-invariant in our setup, it follows that $c(\Omega) = (\sum_{i=1}^r \omega_i)/r$, for any maximal frame $\omega_1, \ldots, \omega_r$. Every face of F is spectral, since spectrality of Ω asserts, for $\omega \in F$, that ω is a convex combination of perfectly distinguishable states, but these states must be in F by the definition of face, and by Proposition 4.2(7) they must be extendable to a frame for F. Next we claim that for each face F of Ω, there is a subgroup of $K = \mathrm{Aut}(\Omega)$, that preserves $\mathrm{span}\, F$ and (necessarily or else it could not consist of automorphisms of Ω) induces automorphisms of F, and acts transitively on the maximal frames in F. This is immediate from strong symmetry, since the maximal frames in F are $|F|$-frames in Ω, and strong symmetry says K can take *any* $|F|$-frame (whether in F or not) to any other. Finally we must show that frames in F are still frames for F viewed in its affine span, but that is so

[6] In fact self-duality does not require spectrality, nor does it require the full strength of strong symmetry: in [53] it was shown that transitivity of $\mathrm{Aut}(\Omega)$ on 2-frames (ordered pairs of perfectly distinguishable states) implies self-duality.

because the cone is perfect, and when the dual cone is represented internally via the self-dualizing inner product, the distinguishing effects are the states themselves (cf. item 4 of Proposition 4.2). Perfection of the cone V_+ implies that the cone over F is self-dual in its linear span according to the restriction of the inner product, so these effects are still in the relative dual cone of F.

In fact, an element of K takes maximal frames of F to maximal frames of F if, and only if, it belongs to the subgroup $K_{\operatorname{span} F}$, that preserves $\operatorname{span} F$. The action of this group on $\operatorname{span} F$ gives a faithful representation of the group $K_{\operatorname{span} F}/K^{\operatorname{span} F}$ (equivalently K_F/K^F), where $K^{\operatorname{span} F}$ is the group of elements in K fixing $\operatorname{span} F$ pointwise. Since we have shown that F is spectral and strongly symmetric, it follows from the claim in the first sentence of this proof that the barycenter of F is $\sum_{i=1}^{|F|} \omega_i/|F|$ for any maximal frame ω_i for F. Any automorphism of F preserves its barycenter, so the automorphism of F induced by any element of K_F must do so, i.e. $K_F \subseteq K^{c(F)}$. Furthermore, $K^{c(F)} \subseteq K_F$. To see this, note that $c(F)$ is in the relative interior of F and therefore $F = \operatorname{Face}(c(F))$. Hence for $\phi \in K^{c(F)}$ we have $\phi(F) = \phi(\operatorname{Face}(c(f))) = \operatorname{Face}(\phi(c(F))) = \operatorname{Face}(c(F)) = F$, i.e. $\phi \in K_F$. Here the second equality is a general fact about automorphisms ϕ (that $\operatorname{Face}(\phi(x)) = \phi(\operatorname{Face}(x))$), and the third is from the assumption $\phi \in K^{c(F)}$. ■

5. Flags and regularity

In this section we provide some further technical tools which will help us to prove our main theorem.

Definition 5.1. Let Ω be a convex set.

(i) A *flag* of Ω is a strictly increasing sequence $F_1 \subset \ldots \subset F_k$ of exposed faces of Ω.

(ii) Ω is called *regular*, if $\operatorname{Aut}(\Omega)$ acts transitively on the set of maximal flags of Ω.

(iii) A subset $\Delta \subseteq \Omega$ is called a *fundamental region* with respect to $\operatorname{Aut}(\Omega)$ if $\Omega = \operatorname{Aut}(\Omega)\Delta$ and $\operatorname{Aut}(\Omega)$-orbits meet the interior of Δ in at most one point.

Lemma 5.2. *If Ω is spectral and strongly symmetric, then it is regular.*

In order to prove this lemma we first establish:

Lemma 5.3. *Let Ω be a strongly symmetric spectral convex set. Let $\omega_1, \ldots, \omega_r$ be a maximal frame in Ω. The sequence*

$$F_i := \bigvee_{1 \le j \le i} \{\omega_j\}, \quad i \in \{1, \ldots, r\} \tag{4}$$

is a maximal flag. Conversely, let $(F_1, \ldots, F_r)$ be a maximal flag of Ω. Then there exists a maximal frame $\omega_1, \omega_2, \ldots, \omega_r$ such that (4) holds.

In other words, the formula (4) gives a bijection between maximal frames in Ω and maximal flags of Ω.

Proof. Let $X = \omega_1, \ldots, \omega_r$ be a maximal frame. It follows from item 1 of Proposition 4.2 that the initial segments of X generate a sequence of faces, the F_i of (4), each properly contained in the next. This is a flag, which we will call Φ_X. In this sequence, the rank $|F_i|$ of F_i is i. Suppose this flag is not maximal. Then it can be enlarged, either by extending it before F_1, or by extending it after F_r, or by inserting some face G with $F_i \subsetneq G \subsetneq F_{i+1}$. It cannot be extended before $F_1 = \{\omega_1\}$, because the only face below the pure state ω_1 is the improper face $\emptyset$. It must have $F_r = \Omega$ by Proposition 4.2 (7), so it cannot be extended beyond F_r. So there must be an $i \in \{1, \ldots, r\}$ and a face G such that $F_i \subsetneq G \subsetneq F_{i+1}$. By Proposition 4.2 (1) $F_i \subsetneq G \subsetneq F_{i+1}$ implies $|F_i| < |G| < |F_{i+1}|$, which contradicts the fact, observed above, that $|F_i| = i$ and $|F_{i+1}| = i+1$ by construction. Since every way of extending the flag Φ_X is inconsistent with the maximality of the frame X, Φ_X is a maximal flag.

Conversely, suppose $\Phi = F_1, \ldots, F_r$ is a maximal flag. By Proposition 4.2(1) each F_i is the join of (the singletons corresponding to) a frame, whose cardinality is $|F_i|$. We will show ("Claim 1") that $F_1 = \{\omega_1\}$ for some extremal point ω_1, and ("Claim 2") that for each $i \in \{2, \ldots, r\}$, there exists an extremal ω_i, distinguishable from every state in the frame F_{i-1}, such that $F_i = F_{i-1} \vee \omega_i$. It follows from the associativity of join that (4) holds for each i, and since $F_r = \Omega$ for a maximal flag, $\omega_1, \ldots, \omega_r$ generates Ω. Since Ω is generated by a maximal frame, and by Proposition 4.2(1) all frames generating the same face have the same cardinality, $\omega_1, \ldots, \omega_r$ has the same cardinality as a maximal frame, whence it *is* a maximal frame.

To show Claim 2 we use the fact, which is part of Proposition 4.2(7), that if $F \subsetneq G$, any frame for F extends to a frame for G by adjoining a frame for $F' \wedge G$. Consider $F = F_{i-1}, G = F_i$, for $i \in \{2, \ldots, r\}$. The frame for the face $F'_{i-1} \wedge F_i$, whose join with F_{i-1} is F_i, is nonempty because F's containment in G is strict. In fact, were $F' \wedge G = 0$ (i.e. $\emptyset$) then we would have $(F' \wedge G) \vee F = F$, while orthomodularity says $(F' \wedge G) \vee F = G$. Say it is $\sigma_1, \ldots, \sigma_m$, for $m \geq 1$. We show that $m = 1$. If $m > 1$, then we can extend the flag Φ to a flag $\widetilde{\Phi}$ defined by $\widetilde{F_j} := F_j$ for $j \in \{1, \ldots, i-1\}$, $\widetilde{F_j} := \widetilde{F}_{j-1} \vee \sigma_{j-i+1}$ for $j \in \{i, \ldots, i+m-1\}$, and $\widetilde{F_j} := F_{j-m+1}$ for $j \in \{i+m, \ldots, r\}$. For $m > 1$, Φ is a proper subflag of $\widetilde{\Phi}$, contradicting Φ's maximality. So we must have $m = 1$, and $\widetilde{\Phi} = \Phi$.

To show Claim 1, that $F_1 = \{\omega_1\}$ for some extremal ω_1, we use essentially the same argument: there is a frame $\eta_1, .., \eta_{|F_1|}$ for F_1, nonempty because $F_1 \neq \emptyset$, and if $|F_1| \neq 1$, then we can extend the flag by prefixing it with the nonempty sequence of subfaces $(H_i := \bigvee_{k \in 1,..i} \{\eta_k\})_{i \in \{1,\ldots,|F_1|-1\}}$, generated by the initial segments of that frame. Since the flag was maximal, the extension must be impossible, so $|F_1| = 1$, hence $F_1 = \{\eta_1\}$, with η_1 extremal. ∎

Lemma 5.2 follows almost immediately.

Proof of Lemma 5.2. Let $\Phi_1 = F_1, \ldots, F_r$ and $\Phi_2 = G_1, \ldots, G_r$ be two maximal flags of Ω. Then by Lemma 5.3 the faces of Φ_1, resp. Φ_2, are the se-

quences of faces generated by the initial segments of the maximal frames $\omega_1, \ldots, \omega_r$, $\eta_1, \ldots, \eta_r$ respectively, defined by the bijection (4). By strong symmetry, there exists $g \in \mathrm{Aut}(\Omega)$ such that for all $i \in \{1, \ldots, r\}$, $g\omega_i = \eta_i$. It follows that $g\Phi_1 = \Phi_2$. ■

The following result of Farran and Robertson together with the ensuing classification of regular convex compact sets is our key tool in proving Theorem 3.5.

Theorem 5.4 (Farran-Robertson [33]). *Let $\Omega \subseteq E$ be regular and $F_1 \subset \ldots \subset F_r$ be a maximal flag. If c_j is the barycenter of F_j, then the $(r-1)$-simplex $\Delta(\Omega) := \mathrm{Conv}(c_1, \ldots, c_r)$ is a fundamental region for Ω with respect to $\mathrm{Aut}(\Omega)$. Moreover,*

$$\pi(\Omega) := \Omega \cap \mathrm{span}(c_1, \ldots, c_r)$$

is a polytope, called the Farran-Robertson polytope *of Ω.*

An immediate corollary is that $\Omega = K.\pi(\Omega)$. We will need a more general result from [49]. It will allow us to understand the facial structure of Ω by understanding that of $\pi(\Omega)$; we will use it to show that frames in $\pi(\Omega)$ correspond to frames in Ω in the strongly symmetric spectral case. The result is stated near the top of p. 369 of [49].[7]

Proposition 5.5 ([49]; see also [33], Theorem 10, and its proof). *Let Ω be a regular convex compact set with automorphism group K and Farran-Robertson polytope $\pi(\Omega)$. Let F be a face of $\pi(\Omega)$ with centroid $c(F)$, and write $K^{c(F)}$ for the isotropy subgroup of K at $c(F)$. Then the orbit $K^{c(F)}F$ is a face of Ω, which we call H_F, and each face of Ω is of the form gH_F for some face F of $\pi(\Omega)$ and some $g \in K$. Moreover, if $g \in K$ and $F_1, F_2, \ldots, F_r$ is a maximal flag of $\pi(\Omega)$ then $gH_{F_1}, gH_{F_2}, \ldots, gH_{F_r}$ is a maximal flag of Ω, and every maximal flag of Ω arises from a flag of $\pi(\Omega)$ in this way.*

Lemma 5.6. *If Ω is spectral and strongly symmetric, then $\pi(\Omega)$ is a simplex whose vertices form a maximal frame.*

Proof. Let the dimension of Ω be n and its rank be r. Pick a maximal flag (it does not matter which one), $F_1, \ldots, F_r$, and let $\omega_1, \ldots, \omega_r$ be the maximal frame corresponding to it via the bijection (4). Using the description of the barycenters of faces from Proposition 4.2 (5) gives $\Delta' = \triangle(\omega_1, \frac{1}{2}(\omega_1+\omega_2), \ldots, \frac{1}{r-1}(\omega_1+\ldots+\omega_{r-1}))$. Since the barycenters just listed are manifestly linearly independent in E (as must be any set of barycenters of faces of a flag), the linear space L spanned by Δ' is $(r-1)$-dimensional. It is easy to see that $\omega_1, \ldots, \omega_{r-1}$ are a basis for L.

Recall that we extend the action of $\mathrm{SO}(E)$ to an action of $\mathrm{SO}(E)$ on V, which must fix the ray over $c(\Omega)$ (pointwise), and equip V with a corresponding invariant inner product such that this action of $\mathrm{SO}(E) \simeq \mathrm{SO}(n)$ is as a subgroup of $\mathrm{SO}(V) \simeq \mathrm{SO}(n+1)$. By Proposition 4.2(3) this invariant inner product can be

[7] An essentially identical (except for its more restricted premise) theorem is stated for a more restricted setting (the subclass of polar representations occuring within adjoint representations of real semisimple groups) in [33].

chosen to be self-dualizing for the cone V_+ and such that all pure states have unit euclidean norm, and we do so.

Since E is embedded as an affine subspace of V not containing 0, the linear subspace L of E is an affine, but not a linear, subspace of V, affinely generated by $c_1, \ldots, c_{r-1}, c_r$, where $c_r = c(\Omega)$ is the zero of E (but not embedded as the zero of V!). Consequently $L = \widetilde{L} \cap E$, where $\widetilde{L} = \operatorname{span}\{\omega_1, \ldots, \omega_r\} \subseteq V$, whence $\pi(\Omega) = L \cap \Omega = \widetilde{L} \cap \Omega$.

We will show that $\widetilde{L} \cap \Omega$ is the simplex $\triangle(\omega_1, \ldots, \omega_r)$. Recall (Proposition 4.2, item 4) that the frame elements $\omega_1, \ldots, \omega_r$ are orthonormal in V. Therefore they are an orthonormal basis for $\widetilde{L}$. By the self-duality of V_+, $\omega_1, \ldots, \omega_r$ are also on extremal rays of the dual cone with respect to the inner product. So everything in V_+ has nonnegative inner product with each of $\omega_1, \ldots, \omega_r$. These constraints impose in particular that $\widetilde{L} \cap V_+$ lies in the closed positive halfspaces

$$H_i^+ := \{x \in \widetilde{L} \mid \langle \omega_i, x \rangle \geq 0\},$$

$i \in \{1, \ldots, r\}$ of each the hyperplanes $H_i := \{x \in \widetilde{L} \mid \langle \omega_i, x \rangle = 0\}$ in $\widetilde{L}$. These constraints define a polyhedral cone which (using the mutual orthogonality of the ω_i) is identical to the cone over the simplex $\triangle(\omega_1, \ldots, \omega_r)$. Since $\omega_1, \ldots, \omega_r$ are in V_+ and in $\widetilde{L}$, we know that $\widetilde{L} \cap V_+$ contains this cone, and since we have just shown that $\widetilde{L} \cap V_+$ is contained in this cone, we have that $\widetilde{L} \cap V_+$ is equal to it, and hence that $\pi(\Omega) = \widetilde{L} \cap \Omega = \triangle(\omega_1, \ldots, \omega_r)$.

Since the states $\omega_1, \ldots, \omega_r$ are the vertices of the Farran-Robertson polytope of Ω, the faces $F_1, \ldots, F_r$ of the polytope defined by the formula (4) are a maximal flag of that polytope. Then by Proposition 5.5, the faces H_{F_i} of Ω are also a maximal flag of Ω. Since H_{F_i} is $K^{c(F_i)}.F_i$, $c(F_i)$ is the centroid of H_{F_i}. Since $c(F_i) = (1/i)\sum_{j=1}^{i} \omega_i$, H_{F_i} is the face of Ω generated by $\omega_1, \ldots, \omega_i$, i.e.

$$H_{F_i} = \bigvee_{i=1}^{i} \omega_i, \quad i \in \{1, \ldots, r\}. \tag{5}$$

So by Lemma 5.3, $\omega_1, \ldots, \omega_i$ are a frame in Ω, not merely in $\pi(\Omega)$, and $\omega_1, \ldots, \omega_r$ is a maximal frame. ■

6. Proof of the Main Result

In this section we use Lemmas 5.2 and 5.6 together with the classification of regular convex bodies in [49] to prove Theorem 3.5. We begin by describing the classification in [49], giving only sufficient detail for our needs. To do so, we need to define a notion called *symmetric space representation* in [49] and [28]; we shall refer to it as a *symmetric space isotropy representation.*

Definition 6.1. Let K be a compact connected Lie group, $\mathfrak{k}$ its Lie algebra. A representation $\rho : K \to \mathrm{SO}(V)$ is called a *symmetric space isotropy representation* if there are a real semisimple Lie algebra $\mathfrak{h}$ with Cartan decomposition $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{p}$, a Lie algebra isomorphism $A : \mathfrak{k} \to \mathfrak{g}$, and an isomorphism L of linear spaces $V \to \mathfrak{p}$ such that $L \circ d\rho(X) \circ L^{-1}(y) = [A(X), y]$ for all $X \in \mathfrak{k}, y \in \mathfrak{p}$.

In other words, $L \circ d\rho(X) \circ L^{-1} = \mathrm{ad}(A(X))$. The terminology "symmetric space isotropy representation" comes from the fact that the tangent space to the symmetric space H/K is $\mathfrak{p}$, and the canonical action of H on H/K has isotropy subgroup, at any point, isomorphic to K, which acts linearly on the tangent space; this action is isomorphic to the one in Definition 6.1.

A maximal abelian subspace $\mathfrak{a}_0$ of such a representation space $\mathfrak{p}$ (or the image of such a subspace under the isomorphism L of Definition 6.1) is termed a *Cartan subspace.*

What we need to know about the Madden-Robertson classification is summarized in the following.

Proposition 6.2 (Madden-Robertson [49]). *Every regular convex body Ω is either a polytope, or affinely isomorphic to the union of the K-orbit, in an irreducible symmetric space isotropy representation of K, of its Farran-Robertson polytope $\pi(\Omega)$, embedded in the representation space $\mathfrak{p}$ as $\pi(\Omega) = \Omega \cap \mathfrak{a}$, where $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{p}$. The set Ω is determined, up to affine isomorphism, by the representation and the affine isomorphism class of the polytope $\pi(\Omega)$. Conversely every irreducible symmetric space isotropy representation contains full-dimensional regular convex bodies of the form $\Omega = K.\pi(\Omega)$ with $\pi(\Omega) = \Omega \cap \mathfrak{a}$. The list of such representations is given in Tables 2-4 of [49], together with the regular convex bodies Ω thus embedded, specified by giving the polytope $\pi(\Omega)$.*

We will also need the following lemma, which in the main summarizes known, but somewhat dispersed, facts about euclidean Jordan algebras, along with their implications for our concerns.

Lemma 6.3. *A simple euclidean Jordan algebra V with unit e and rank r may be identified with the subspace $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the reductive Lie algebra $\mathfrak{g}$ of $\mathrm{Aut}V_+$. We have $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{p}_0$, and $\mathfrak{p}_0$ is the traceless subspace of V, while $\mathfrak{r} = \mathbb{R}e \simeq \mathbb{R}$. The action of the automorphism group $\mathrm{Aut}_0(\Omega)$ of V's normalized state space Ω on $\mathfrak{p}_0$ is an irreducible symmetric space isotropy representation. Letting $\mathfrak{a}_0$ be a Cartan subspace of $\mathfrak{p}_0$, $\Omega_0 := \Omega - c(\Omega)$, the translation of Ω into $\mathfrak{p}_0$, has Farran-Robertson polytope $\mathfrak{a}_0 \cap \Omega_0$, which is a simplex with r vertices. By identifying $c(\Omega)$ with 0, this linear action is identified with the action of $\mathrm{Aut}(\Omega)$ on $\mathrm{aff}(\Omega_0)$, and the action on Ω identified with that on Ω_0.*

Since the proof of this Lemma mostly references or recapitulates known, but somewhat involved, theory of euclidean Jordan algebras, we have placed it in Appendix A.

Proof of Theorem 3.5. By Lemmas 5.2 and 5.6, every strongly symmetric convex compact set Ω is a regular compact convex set with $\pi(\Omega)$ a simplex. Madden and Robertson [49] provide, in their Tables 1-4, a list of the regular convex compact sets Ω, together with the corresponding polytopes $\pi(\Omega)$. When $\pi(\Omega)$ is a simplex, the following cases occur:

Case 1: Ω is itself a polytope. Then $\Omega = \pi(\Omega)$ so Ω is a simplex.

Case 2: Ω is not a polytope and $\pi(\Omega)$ is a simplex.

A: $\pi(\Omega)$ is an interval, i.e. $\pi(\Omega) = \triangle_1$. Then Proposition 2.7 implies that Ω is a ball, and hence can be realized as the normalized state space of a rank 2 simple euclidean Jordan algebra.

B: $\pi(\Omega) = \triangle_n$ with $n \geq 2$. Comparing the list of Madden and Robertson with the list of simple euclidean Jordan algebras shows that the regular convex body Ω is always affinely isomorphic to the normalized state space of a simple euclidean Jordan algebra. We give a more formal account of this comparison, using Proposition 6.2 and Lemma 6.3, below.

Lemma 6.3 exhibits the normalized state space Ω, which is a strongly symmetric compact convex set, as affinely isomorphic to the K-orbit Ω_0 of a Farran-Robertson polytope $\Omega_0 \cap \mathfrak{a}_0$ in a Cartan subspace $\mathfrak{a}_0$ of a symmetric space isotropy representation, which is moreover a simplex. Since by Proposition 6.2 the representation and the affine isomorphism class of $\pi(\Omega)$ determine the regular convex body Ω, we see that the normalized EJA state spaces Ω are affinely isomorphic to the regular convex bodies Ω in those same symmetric space isotropy representations in Tables 2-4 of [49]. Tables 2 and 4 of [49] present the representations by giving noncompact forms H/K of the symmetric space; Table 3 of [49] gives the pair $(\mathfrak{h}, \mathfrak{k})$. Examination of those tables shows that the representations coming from simple EJAs in this manner—given explicitly in Table 1 above by the pair $\mathfrak{g}$ (playing the role of $\mathfrak{h}$ in Definition 6.1) and $\mathfrak{k}$—exhaust the the nonpolytopal regular Ω with $\pi(\Omega)$ a simplex Δ_n, $n \geq 2$, so the higher-rank regular convex compact sets with $\pi(\Omega)$ a simplex are precisely the normalized EJA state spaces, completing the proof of Theorem 3.5. ■

To base the proof of Theorem 3.5 on the classification is not completely satisfactory. It would be desirable to have an intrinsic proof.

While it is not needed for our proof, a little more detail of the Madden-Robertson classification and its mathematical underpinnings may be of interest. In a symmetric space isotropy representation, instances of $\pi(\Omega)$ in $\mathfrak{a}$ occur as the convex hulls of orbits of particular elements of $\mathfrak{a}$ under the action of the Weyl group $W := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ (which is a finite reflection group), namely fundamental weights dual to roots that are at either end of the Coxeter diagram for the root system of K. Coxeter diagrams are like Dynkin diagrams, but without information about the root length, which is irrelevant for the purpose of specifying a finite reflection group. This description of the polytopes $\pi(\Omega)$ is obtained by using the fact that the polytope $\pi(\Omega)$ is regular, and Coxeter's characterization (see for example [27]) of the ways in which regular polytopes can occur as convex hulls of orbits in finite reflection groups.

The fact that regular convex bodies appear as convex hulls of orbits in symmetric space isotropy representations is derived by first showing that they are convex hulls of orbits in polar representations of connected compact groups, and then using Dadok's classification [28] of such representations, which shows that they are orbit-equivalent to symmetric space isotropy representations. A polar representation is one in which there exists a subspace that meets every orbit orthogonally. In symmetric space isotropy representations, this is the Cartan subspace $\mathfrak{a}$.

7. Implications of our result for general probabilistic theories

As mentioned in the introduction, Theorem 3.5 may be viewed as purely a matter of convex geometry, stating that geometrically natural spectrality and symmetry assumptions for a compact convex set imply Jordan-algebraic structure. On the other hand, these conditions make use of the notion of perfect distinguishability, which, although it is geometrically natural, originated in the context of general probabilistic theories, and is closely related to information processing properties and protocols in such theories. While the convex sets of normalized states are fundamental to describing systems in the GPT framework, additional structure is often specified, for example a distinguished convex semigroup of positive maps, on the ordered linear space (V, Ω, u), representing possible dynamical processes, and the analysis may go beyond the structure of single systems by specifying ways of combining two or more systems into a composite system, allowing for the study of correlation, entanglement, and related phenomena.

In particular, in GPTs one often allows the possibility of describing a system by specifying not only a convex compact set of normalized states, but also a convex compact *subset* $\mathcal{E}^{allowed}$ of the set $\mathcal{E}$ of effects, satisfying the natural condition $e \in \mathcal{E}^{allowed} \implies (u - e) \in \mathcal{E}^{allowed}$. In such a setting, one could define perfect distinguishability and frames as we have done, except with measurements restricted to those consisting of effects in $\mathcal{E}^{allowed}$. In principle, fewer sets of states might be distinguishable in this situation, leading to fewer frames, which—if strong symmetry or spectrality were defined in terms of this smaller set of frames—could lead to state spaces that are strongly symmetric, or spectral, with respect to $\mathcal{E}$, failing to be so with respect to $\mathcal{E}^{allowed}$. In fact, this is how strong symmetry and spectrality were defined in [13], but there it was also shown that the conjunction of strong symmetry and spectrality, even with respect to this definition that depends on a choice of both on Ω and $\mathcal{E}^{allowed} \subseteq \mathcal{E}$, implies that $\mathcal{E}^{allowed} = \mathcal{E}$, so that the conjunction of strong symmetry and spectrality in the sense of [13] is the same as in our sense, which is what allows us to use the theorems in [13] that concern consequences of this conjunction. The condition $\mathcal{E}^{allowed} = \mathcal{E}$ is frequently used in the GPT literature, where it is usually called the *no-restriction property*.

GPT theory has roots in the quantum logic program initiated by Birkhoff and von Neumann [21], in which the convex compact set of normalized states on various "logical" objects abstracted from structures on quantum systems, such as modular or orthomodular posets or lattices, often plays an important role. The work of Mackey [48] was an important catalyst to rapid and extensive further development, during the 1960s and early 1970s, of both quantum logic based and more purely convexity based descriptions of abstract physical systems. We mention in particular the work of Foulis and Randall [34, 35, 37, 36] and of Ludwig and his group [47]. This work, especially that of Ludwig, was connected with the development of the concepts of base norm and order unit spaces in functional analysis, a connection explored primarily in the quantum case in [30], and in greater generality in [31].

While these lines of work continued into the 1990s, there was a revival of interest, especially among physicists, in the area around the turn of the millenium, driven primarily by the growth of the field of quantum information processing, which raised questions about the conceptual underpinnings of the peculiar nature of quantum in-

formation that are particularly suited to investigation in a broader setting such as that of GPTs, focused on probabilistic properties, often relatively abstract, of systems and theories that are especially relevant to the nature of information. Hardy's [39, 40] axiomatic derivation of the complex quantum formalism, was particularly important in sparking this revival of interest. The work of Popescu and Rohrlich [56] was also important: they asked why the correlations between observables on distinct, but entangled, quantum systems, that are nonclassical by virtue of violating Bell-type inequalities, nevertheless, by virtue of Tsirel'son's bound [58], do not violate these inequalities to the maximum degree permitted by probability theory constrained only by the requirement that the choice of measurement on one system does not affect the marginal probabilities on the other system—a "no-signaling" requirement that is incorporated into most notions of composites in the GPT framework. While their work was not explicitly in a GPT setting, it can be viewed in those terms, because such correlations can be realized in a GPT composite of nonclassical, but also non-quantum, state spaces: the square bits of Example 2.6. Given the importance of entanglement to many of the phenomena and protocols of quantum information, this resurgence of interest involved increased attention to composite systems and entanglement, as well as a focus on implications for information-processing and computation generally. By 2005, Barrett had coined the term "general probabilistic theories" in the paper [16]. In this new wave of research, the setting is often finite-dimensional, since quantum information and computation protocols are often formulated in finite dimension—in particular, n qubits have a Hilbert space of dimension 2^n, and associated state space Ω of density matrices of dimension $2^{2n}-1$. Most of the conceptual questions about the nature of information, and the possibility of various kinds of information processing protocols or physical phenomena, that arise from this point of view are just as salient in finite dimension as in the infinite dimensional setting, and more tractable.

In the remainder of this section, we describe some implications of our result for two such areas: thermodynamics and its statisical-mechanical underpinning, and computational query complexity. See [10] for more detail. Important aspects of quantum and classical thermodynamics and of query complexity have been generalized to classes of GPTs satisfying natural postulates including or implying spectrality and strong symmetry; however, our result shows that these apply to a narrower class of theories than might have been hoped, already close to complex quantum theory since their state spaces are Jordan-algebraic.

In [46] it was shown that five GPT principles, of which the fifth is strong symmetry, allow the formulation of a reasonable query model generalizing the quantum one, and imply that to have probability $1/2$ or greater of correctly identifying a marked item in a list of N items in Grover's search problem, the number of queries to the list must be at least $(\frac{3}{2}-\sqrt{2})\sqrt{N/k}$, where k is the maximal order of interference of the GPT theory. A lower bound of $\Omega(\sqrt{N})$ (meaning there exists a constant c such that the number of queries is greater than or equal to $c\sqrt{N}$) was established in the quantum case in [20]; it is achieved by Grover's algorithm, and improves on the classical lower bound of $\Omega(N)$. The GPT bound in [46] is also $\Omega(\sqrt{N})$. This limits the potential gain from higher-order interference of degree k in this setting to at most a constant factor, $c/\sqrt{k}$ compared to quantum. However, it can be shown

(cf. [23]) that the conjunction of the principles used in [46] implies spectrality. Together with Theorem 3.5 this implies that the GPT systems considered in [46] are Jordan-algebraic, and hence [59, 54] that higher-order interference ($k > 2$) is not possible in this setting. In [12], a definition of query computation was formulated and two results were obtained concerning query computation in GPTs under nearly the same assumptions as in [46]; Theorem 3.5 limits these results, too, to simple Jordan-algebraic systems and classical systems.

In [45], spectrality and strong symmetry were used to extend important results in classical and quantum thermodynamics to the context of strongly symmetric spectral systems. These results use the mathematical notions of *majorization* and *Schur concavity*. An l-tuple $\lambda = (\lambda_1, \ldots, \lambda_l)$ of real numbers majorizes an m-tuple $\sigma = (\sigma_1, \ldots, \sigma_m)$ of real numbers if, when the shorter tuple is extended to have length $n := \max(l, m)$ by appending 0's, yielding $\overline{\lambda}$ and $\overline{\sigma}$, and the two are rearranged in monotonically decreasing order, yielding $\overline{\lambda}^{\downarrow}$ and $\overline{\sigma}^{\downarrow}$, we have that for each $k \in \{1, \ldots, n\}$, $\sum_{i=1}^{k} \overline{\lambda_i}^{\downarrow} \geq \overline{\sigma_i}^{\downarrow}$. A function $f : \cup_{n \in \mathbb{N}} \mathbb{R}^n \to \mathbb{R}$ is called *Schur concave* if whenever x majorizes y, $f(y) \geq f(x)$. "x majorizes y" is generally interpreted as a formalization of the idea that x is "more mixed" or "more random" than y, because of the Birkhoff-von Neumann theorem, which states (in the case where x and y have equal length) that x majorizing y is equivalent to y being a convex combination of vectors obtained from x by permuting its entries. Schur concave functions are often viewed as real-valued "measures of randomness" or "generalized entropies," since they are precisely the real-valued functions that can never decrease under such operations. This fact is used in applying majorization to microcanonical classical thermodynamics (cf. e.g. [1]).

Given spectrality, it is natural to investigate majorization and Schur concave functions on the spectra (with multiplicity) of states. For each Schur concave function f, one defines a corresponding generalized entropy on the state space, as giving the value of f on the spectrum of the state. For example, the von Neumann entropy of a quantum state, which is given by the Shannon entropy $H(p) := -\sum_i p_i \ln p_i$ of its spectrum $p = \{p_1, \ldots, p_n\}$, is one such entropy. In general theories, one can define the *measurement entropy* and the *preparation entropy* of states. The measurement entropy of state σ is the minimum, over finegrained measurements,[8] of the Shannon entropy of the probabilities of the outcomes when the measurement is made on a system in state σ; the preparation entropy is the minimum entropy of probabilities p_i such that $\sigma = \sum_i p_i \omega_i$, for pure states ω_i. Analogous definitions can also be made for the generalized entropies determined by Schur concave functions other than Shannon entropy. In quantum theory, the preparation and measurement entropies corresponding to a given f are equal to each other and to the spectral entropy corresponding to f.

In [45], it was shown in the GPT context, that assuming spectrality and strong symmetry, the outcome probabilities of any fine-grained measurement on σ are majorized by those of the spectral measurement (which are equal to σ's spectrum), and hence that the measurement entropy determined by any Schur concave function is

[8] A measurement is called fine-grained if no effect in the measurement has a decomposition as a strictly positive linear combination of distinct effects that are not multiples of each other.

equal to the corresponding spectral entropy.[9] In parallel work in [24] the same conclusion was obtained in the setting of GPTs using the properties (defined therein) of causality, purification, purity preservation under both parallel and sequential composition of pure operations, and strong symmetry. The first four of these assumptions together (see [24] and references therein for definitions) can be shown to imply spectrality. So in light of Theorem 3.5, the setting of [45, 24] is no more general than that of simple Jordan-algebraic state spaces, and classical ones. The same conclusions were also obtained from a somewhat different set of premises, defining what are called *sharp theories with purification*, in [23] and [25]. However in [11] it was shown that systems in such theories are also Jordan-algebraic (although the class of systems is not precisely simple Jordan algebras and classical theory, since some nonclassical nonsimple state spaces are definitely allowed, and to the best of our knowledge it is not known whether all simple Jordan algebras are). In [6] the same conclusions were obtained from projectivity of the state space and symmetry of transition probabilities (equivalently, projectivity and self-duality of the state cone, which are in turn equivalent ([4], cf. also [6]) to its perfection together with the normalization of the orthogonal projections onto the linear spans of faces). All Jordan-algebraic state spaces have these properties, but it is an open question whether they are the only ones.

8. Comparison to other characterizations of classes of Jordan-algebraic state spaces

There are various characterizations of classes of Jordan-algebraic state spaces from convexity and/or symmetry properties, but differing significantly from ours; in this section we review some of these. Most of these characterize the class of all Jordan-algebraic state spaces, rather than just the simple ones and the simplicial ones. Some of them characterize generalizations of euclidean Jordan algebras that extend to infinite dimensional cases: the JB-algebras (which includes the self-adjoint parts of C^* algebras) and the JBW-algebras (JB-algebras with predual, including the self-adjoint parts of von Neumann (also known as W^*) algebras). In this section, we consider two main characterizations: one due to Koecher and Vinberg, involving homogeneity and self-duality of the cone V_+, and one due to Alfsen and Shultz, involving a special class of positive projections associated with the faces of V_+, as well as additional properties.

One of the most important such characterizations is the Koecher-Vinberg theorem [43, 60], which states that the finite-dimensional homogeneous self-dual cones are precisely the Jordan-algebraic ones. A (closed, pointed, generating) cone is *homogeneous* if its group of affine automorphisms acts transitively on its interior. Since all bases of such a cone are affinely isomorphic, it follows that a state space Ω such that $\mathbb{R}_+\Omega$ is homogeneous and self-dual, is the normal state space of a euclidean Jordan algebra. Since, as noted in Proposition 4.2, spectrality and strong symmetry imply self-duality of the cone over the normalized state space, showing directly that spectrality and strong symmetry imply homogeneity would give an

[9] The theorem was stated for the Renyi α-entropies, but the proof uses Schur concavity and applies to arbitrary Schur concave functions.

alternative proof of our main theorem. In [61], formulated in a variant of the GPT framework that uses the formalism of [34, 35], assumptions are described that, nontrivially, imply homogeneity and self-duality, giving rise to Jordan structure via the Koecher-Vinberg theorem. Bellissard and Iochum [17] generalized the Koecher-Vinberg theorem to include a class of infinite-dimensional cases: they characterized the facially homogeneous self-dual cones in real Hilbert spaces as the positive cones of JBW-algebras. A *facially homogeneous* self-dual cone V_+ in a real Hilbert space V (the notion is due to Connes [26]) is one for which, for any face F, the difference $P_F - P_{F^\perp}$ between the orthogonal (with respect to the self-dualizing inner product) projections P_F and $P_{F^\perp}$ onto the span of F and the span of

$$F^\perp := \{x \in K : \langle F, x\rangle = 0\rangle\},$$

respectively, belongs to the Lie algebra of $\mathrm{Aut}(V_+)$. Facial homogeneity is known to coincide with homogeneity in finite dimension [18], and indeed for JB algebras with finite faithful normal trace [19] (this class does not, however, encompass all the JBW algebras).

Another important result is due to Alfsen and Shultz who characterized the positive cones of JB algebras and of JBW algebras [2] (see also [3]). It uses their notion of *compression*: in terms of the formalism of embedded state spaces (V, Ω, u), this is a positive idempotent linear map $P : V^* \to V^*$ which is *normalized* and *bicomplemented*. An idempotent that is positive with respect to a cone K is *complemented* if there exists projection P', called its *complement*, such that $\mathrm{im}(P) \cap K = \ker(P') \cap K$ and $\ker(P) \cap K = \mathrm{im}(P') \cap K$, and *bicomplemented* if both P and P^* are complemented (the latter with respect to the cone K^*). An idempotent $P : V_+^* \to V_+^*$ is *normalized* if $P^* u \leq u$, where u is the order unit in V_+^*. In [13], the dual $P^* : V \to V$ of a compression $P : V^* \to V^*$ was termed a *filter*. A state space is said to be *projective* if every face of V_+^* is the positive part $\mathrm{im} P \cap V_+^*$ of the image of a compression P (equivalently, every face of V_+ is the positive image of a filter). Projectivity of a finite-dimensional state space implies that the lattice of faces of V_+^* is orthomodular and atomic, and that there is a bijection $\gamma : e \mapsto \gamma(e)$ between the atomic effects $P^* u$ (where P^* is the compression onto an atomic face) and the pure states: $\langle \gamma(e), e\rangle = 1$, and $\gamma(e)$ is the unique pure state giving unit probability to e; likewise $\gamma^{-1}(\omega)$ is the unique atomic effect given unit probability by ω.

The finite-dimensional case of Alfsen and Shultz' result is then that an embedded state space (V, Ω) is the normalized state space of an EJA if and only if it satisfies (1) *projectivity*, which states that every face of V_+^* is the positive part of the image of a filter, (2) *symmetry of transition probabilities*, which states that for any two atomic effects e and f $\langle e, \gamma(f)\rangle = \langle f, \gamma(e)\rangle$, and (3) *purity preservation*, which states that filters take pure states in Ω to multiples of pure states. Araki [4] obtained a similar result in finite dimension, using projectivity with respect to a notion of filter *prima facie* somewhat more general than the dual of a compression, but equivalent in the context of symmetry of transition probabilities and purity-preservation which he also assumed.

From Proposition 4.2, the conjunction of spectrality and strong symmetry implies projectivity and self-duality of V_+. Since self-duality near-trivially implies symmetry of transition probabilities [13], a direct argument from spectrality and

strong symmetry to purity preservation by filters would give another proof of the Jordan structure, using the Alfsen-Shultz characterization. In fact, the result in [13] was obtained by showing that filters preserve purity—but the additional assumption of absence of higher-order interference was used in showing this.

Alternative properties characterizing the Jordan-algebraic state spaces among those satisfying projectivity and symmetry of transition probabilities are (1) the Hilbert ball property, or (2) the satisfaction of the atomic covering law by the lattice of faces of the state space. A finite-dimensional convex set has Alfsen and Shultz' *Hilbert ball property* if and only if for every pair of extreme points of Ω, the face they generate is affinely isomorphic to a euclidean ball. (See [3], Def. 9.9, for additional technical conditions relevant in infinite dimension.) The *atomic covering* law for a lower-bounded lattice states that if a is an atom in the lattice, and b any element of the lattice, then either $a \vee b = b$, or $a \vee b$ covers b. Here "x *covers* y" means $x > y$ and there exists no w such that $x > w > y$, i.e. x is above y and there is nothing between them, and an *atom* is an element that covers 0. So an alternative proof of our result could also aim at establishing either one of these properties, or the absence of higher order interference, directly from spectrality and strong symmetry.

9. Characterizing the spaces of complex hermitian density matrices within the class of simple Jordan-algebraic state spaces and simplices

As mentioned in the introduction, characterizations of classes of Jordan state spaces using properties abstracted from the usual (complex hermitian) quantum density matrices are often motivated by attempts to characterize the quantum density matrices themselves. Such characterizations of quantum state space, whether in terms of postulates whose appeal is mathematical, physical, informational, or some combination of these, often proceed by first characterizing Jordan-algebraic state spaces, or some subset thereof, and then adding an additional postulate or set of postulates that narrows things down to standard, i.e. complex, quantum theory. In this section we describe two important classes of such postulates: the first class involves relationships between the observables of a system and the generators of possible reversible dynamical evolutions, while the second class involves the possibility of combining systems in a *tomographically local* way, meaning that the statistical correlations between local observables on the components of a composite system determine its state completely. Either of these can, of course, be used in conjunction with Theorem 3.5 to characterize the irreducible complex quantum systems. (Depending on the details of the postulate used, it may be necessary to add an additional assumption to rule out "classical" state spaces, i.e. the simplices.)

Given the no-restriction hypothesis, the simplices, which are interpreted as the state spaces of classical systems, may be ruled out by many alternative natural postulates, many of which have interpretations in terms of information-processing characteristics of systems, in the GPT framework. With no-restriction, these are properties purely of the convex compact set Ω. Among these alternatives we mention: existence of a tradeoff between information gained about an unknown state, and disturbance to that state (a result reported in [16]); impossibility of universal cloning, or of universal broadcasting [7, 8], the existence of a state having two different convex

decompositions into pure states (a more or less folkloric mathematical fact that is the finite-dimensional case of Choquet's theorem); the lack of universal compatibility of measurements [55]; and nontriviality of the connected identity component of the automorphism group of the normalized states (emphasized by Hardy [39, 40]). If one uses such an assumption to rule out the simplices, one may then concentrate on excluding the normalized state spaces of the simple Jordan algebras other than the complex hermitian ones.

The first such assumption we will consider is that of energy observability, introduced for this purpose in [13]. (Unlike some of the other assumptions we will consider, it also rules out the simplices, since it implies nontriviality of $\mathrm{Aut}_0(\Omega)$.)

Definition 9.1. A normalized state space Ω is said to have *energy observability* ([13], Def. 30) if the Lie algebra $\mathfrak{aut}(\Omega)$ of $\mathrm{Aut}(\Omega)$ is nontrivial and there exists an injective linear map ϕ from $\mathfrak{aut}(\Omega)$ to the observable space V^* of the system, such that for each $x \in \mathfrak{aut}(\Omega)$, $\phi(x)$ is conserved by the one-parameter subgroup generated by x, and $\phi(x) = \lambda u$ (for some $\lambda \in \mathbb{R}$) if and only if $x = 0$.

Proposition 9.2 ([13])**.** *Let Ω be the normalized state space of a simple Jordan algebra, satisfying energy observability. Then there is an n such that Ω is affinely isomorphic to the normalized state space of the Jordan algebra $M_n(\mathbb{C})_{sa}$, i.e. the set of density matrices of a finite-dimensional quantum system.*

Note that energy observability is formulated in the convex framework without reference to Jordan structure. The terminology is motivated by the idea that a continuous one-parameter subgroup of automorphisms is a potential reversible dynamical time-evolution, and in quantum physics the generator of such an evolution is a Hermitian operator H (the Hamiltonian) conserved by the evolution (identified with energy). Here "generator" is meant in the "physicists" sense that the evolution operator is $\omega \mapsto e^{iHt}$.[10] The assumption that $\mathfrak{aut}\ (\Omega)$ is nontrivial is there because without it there is nothing that fits the intuitive notion of energy that inspired the definition, although it should also be noted that it rules out the simplices.

The notion of energy observability is closely related to, and inspired by, Alfsen and Shultz' notion of dynamical correspondence, and Connes' notion of orientation, on a Jordan algebra. However, these notions are not purely convex, but also involve the Jordan structure. To formulate the notion of dynamical correspondence, one first needs to note that for a JB-algebra V, the Lie algebra $\mathfrak{g}$ of $\mathrm{Aut}(V_+)$ (the elements of which are called order-derivations) decomposes orthogonally into spaces $\mathfrak{k}$ of skew-adjoint, and $\mathfrak{p}$ of self-adjoint (with respect to the inner product derived from the Killing form), order derivations—this is the Cartan decomposition of $\mathfrak{g}$, in the finite-dimensional case. The self-adjoint ones may be identified with the space of Jordan multiplication operators, $L_a : b \mapsto a \bullet b$. The *skew order-derivations* are precisely the generators of one-parameter groups of automorphisms of the Jordan algebra (cf. Lemma 2.81 of [3]), hence they are (linearized extensions of) generators

[10] In the usual mathematical terminology, the generator of this evolution is instead the anti-Hermitian operator iH; then the injection from the Lie algebra of generators of one-parameter subgroups of automorphisms (i.e. $\mathfrak{lie}(\mathrm{Aut}(\Omega)) \equiv \mathfrak{aut}\ (\Omega)$) into the observables is just $X \mapsto -iX$.

of one-parameter groups of affine automorphisms of A's normalized state space (cf. [32]). (In finite dimension, they are *precisely* the generators of such one-parameter groups.)

A *dynamical correspondence* ψ on a JB-algebra A is a linear map, $\psi : a \mapsto \psi_a$, of A into the set of skew order-derivations of A, such that (1) $[\psi_a, \psi_b] = -[L_a, L_b]$, and (2) $\psi_a a = 0$. Although the map $a \mapsto \psi_a$ (which is not explicitly required to be either injective or surjective) is in the opposite direction from the injection of generators into the space of observables required by energy observability, the condition $\psi_a a = 0$ is otherwise the analogue of the conservation condition in the definition of energy observability: it says that a is conserved by the evolution generated by ψ_a. Condition (1) is motivated by the fact mentioned above, that in the Jordan case A itself may be identified with a subspace, $\mathfrak{p}$, of the Lie algebra of automorphisms of A_+, since the Jordan multiplication operators act as elements of $\mathfrak{p}$. The condition says that the map $a \mapsto \psi_a$ commutes with the Lie bracket, up to a minus sign. Such a minus sign would arise if the map $a \mapsto \psi_a$ implemented multiplication by a complex unit i on $\mathfrak{g}$. A *Connes orientation* ([3], Definition 6.8) on a JBW-algebra is a complex structure I (so, $I^2 = -1$) on $\mathfrak{g}/Z(\mathfrak{g})$, compatible with Lie brackets and the Cartan involution $\dagger$ in the sense that $[Ix, y] = [x, Iy] = I[x, y]$ and $I(x^\dagger) = -(Ix)^\dagger$. (Connes' notion was originally developed in the case in which A_+ is a facially homogeneous self-dual cone in a Hilbert space.) Connes' orientations are indeed in bijection with dynamical correspondences on JBW-algebras ([3], Theorem 6.18).

By Proposition 10.27 of [3], a dynamical correspondence on a JB-algebra A determines a unique C^*-algebra structure on $A + iA$, such that A is the self-adjoint part and A's Jordan product is the symmetrized C^*-product. In the special case in which A is a JBW-algebra the product on $A + iA$ is in fact a W^*-product, and $A + iA$ is a W^*, i.e. von Neumann, algebra. It should not be surprising that such a complex structure characterizes the self-adjoint parts of complex $*$-algebras in the finite-dimensional case, in light of the fact, evident from Table 1, that the only family of Lie algebras $\mathfrak{g} = \mathfrak{aut}(V_+)$ of simple EJAs whose semisimple part (which is $\mathfrak{g}/Z(\mathfrak{g})$ in that case) is a complex Lie algebra, is $\mathfrak{sl}(m, \mathbb{C})$, corresponding to the Jordan algebra $V = M_m(\mathbb{C})_{sa}$.

A different approach to ruling out the Jordan-algebraic systems other than complex quantum theory involves introducing an appropriate notion of composite system consisting of two or more "subsystems". The existence of tomographically local Jordan-algebraic composites of Jordan-algebraic systems can then be used as a postulate to narrow things down to complex quantum systems. Tomographic locality can be mathematically formulated as the requirement that the ambient vector space V_{AB} spanned by the cone of unnormalized states of a composite AB of systems A, B whose ambient vector spaces are V_A and V_B, be the real tensor product $V_A \otimes V_B$.[11] In [15] it was shown, building on work of H. Hanche-Olsen

[11]The notion of "tomography" in this context is that of determining the state of a system by making various measurements on identically prepared copies of a system. "Local" tomography of a composite system is possible if one can estimate the state by making measurements on its parts, A and B, and estimating the correlations between sufficiently many measurement results. If $V_{AB} = V_A \otimes V_B$, then products of effects $e_A \otimes f_B$ span the dual of the state space, so determining the probabilities of a spanning set allows one to determine the components of the state in a basis.

[38], that the existence of a tomographically local Jordan-algebraic composite of a Jordan-algebraic system A with a qubit (the lowest-dimensional nontrivial complex quantum system, whose state space is a three-dimensional ball), satisfying some other natural desiderata, implies that A must be complex quantum. However, this does not rule out the possibility of tomographically local theories whose systems are spectral and strongly symmetric but in which qubits do not occur as a system type that must be composable with other systems.

In [50], Ll. Masanes and M. Müller formulated five postulates, one of which is tomographic locality, applicable to theories whose systems are described in the GPT framework, and showed that the only theories satisfying them are finite-dimensional complex quantum theory and finite-dimensional classical theory. Their notion of theory is not quite fully explicit, but it appears to require that for any two systems of the theory, there exists another system of the theory that is a locally tomographic composite of those systems. Since the four postulates not referring to composite systems are satisfied by all systems (satisfying the no-restriction property) that either have simple Jordan-algebraic state spaces, or whose state spaces are simplices, we can combine their result with the main result of this paper to conclude that any collection of (finite-dimensional) systems satisfying strong symmetry and spectrality, and closed under the formation of tomographically local composites, must consist either entirely of complex quantum systems, or entirely of classical systems.

That the tomographic locality of the assumed composites is necessary for these results is indicated, for example, by the constructions in [9] of theories in which some of the Jordan-algebraic systems other than complex quantum ones can be combined to form composites that are not tomographically local.

A similar argument involving tomographic locality can be made using the result of [51], in which it is shown that only for $d = 3$ does there exist a composite, satisfying tomographic locality and continuous reversible transitivity on pure states, of two systems each of which has a euclidean d-ball as state space. Since by Proposition 2.7 and Theorem 3.5, spectrality and strong symmetry imply that *bits* (i.e. systems whose largest frame is of cardinality 2) are balls, and also that nonclassical systems are simple Jordan-algebraic and hence have continuous reversible transitivity on pure states, it follows from Theorem 3.5 and [51] that no tomographically local composite of a bit with itself can preserve spectrality and strong symmetry, except in the complex quantum case (for which bits are 3-balls) or the classical case.

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A. Higher-rank euclidean Jordan algebras in the Madden-Robertson classification

In this appendix we give the proof of Lemma 6.3, which we here restate.

Lemma A.1. *A simple euclidean Jordan algebra V may be identified with the subspace $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the reductive Lie algebra $\mathfrak{g}$ of $\mathrm{Aut}(V_+)$. We have $\mathfrak{p} = \mathbb{R} \oplus \mathfrak{p}_0$, and $\mathfrak{p}_0$ is the traceless subspace of V. The action of the connected identity component $\mathrm{Aut}_0(\Omega)$ of the automorphism group of V's normalized state space Ω on the traceless subspace is an irreducible symmetric space isotropy representation. $\Omega_0 := \Omega - c(\Omega)$, the translation of Ω into the traceless subspace, has Farran-Robertson polytope $\mathfrak{a}_0 \cap \Omega_0$, which is a simplex. By identifying $c(\Omega)$ with 0, this linear action is identified with the action of $\mathrm{Aut}(\Omega)$ on $\mathrm{aff}(\Omega)$, and the action on Ω identified with that on Ω_0.*

Proof of Lemma 6.3. The affine automorphism group $G := \mathrm{Aut}(V_+)$ of the cone of squares in an EJA V, like the automorphism group of any self-dual cone [52], is reductive. If the algebra is simple, then $\mathrm{Aut}(V_+) = G^s \times \mathbb{R}_+$, where G^s is simple. V is an irreducible *spherical* representation space for the semisimple group G^s (cf. e.g. [57], p. 42) which means that G^s's maximal compact subgroups, and therefore also their connected identity components, have one-dimensional fixed-point spaces. (It is also irreducible for G, which has the same compact subgroups as G^s.) We may choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{k} = \mathfrak{lie}(K)$, where K is the connected identity component of the maximal compact subgroup of G^s (and therefore also of G) that fixes the identity e. Note that this is a Cartan decomposition of the Lie algebra of the *reductive* group G.[12] In fact we have $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{r}$, where $\mathfrak{r} \simeq \mathbb{R}$ is the Lie algebra of the $\mathbb{R}_+$ factor in G, and $\mathfrak{g}^s = \mathfrak{k} \oplus \mathfrak{p}_0$ the Cartan decomposition of the simple Lie algebra $\mathfrak{g}^s$.

The Jordan algebra V itself can be identified with $\mathfrak{p}$, via the linear bijection $x \mapsto x.e$ for $x \in \mathfrak{p}$. [13] Here the linear action of x on V is an element of $\mathfrak{g}$, which acts on V via the differential of the linear action of $\mathrm{Aut}(V_+)$ on V. In the orthogonal (with respect to the canonical inner product on the Jordan algebra, which agrees up to constants with that derived from the form on $\mathfrak{g}$) decomposition $V \simeq \mathfrak{p} = \mathbb{R} \oplus \mathfrak{p}_0$, $\mathbb{R}$ is the span $\mathbb{R}e$ of the Jordan unit e. We can expand $x \in V$ as $x = x_e e + x_0$, with $x_e \in \mathbb{R}$; the coefficient x_e is $(\mathrm{tr}x)/\mathrm{rank}V$, and x_0 is of course traceless.

Although the actions of $G = \mathrm{Aut}(V_+)$, and of G^s, on V are not their adjoint actions (restricted to $\mathfrak{p}$), the restriction of these actions to K does coincide (on $\mathfrak{p}$) with the restriction of the adjoint action. In other words, $K \curvearrowright \mathfrak{p}_0$ is a symmetric space isotropy representation (Definition 6.1), of the connected compact group K, associated with the noncompact irreducible symmetric space $(G^s)_0/K$. This can be checked explicitly, or deduced from general considerations. For example, in the

[12] See [44], Ch. VII §2 for a definition of reductive group and the extension of the notion of Cartan decomposition to such groups; or see the introduction and Appendix of [22] for the more straightforward case of linear groups (which includes $G = \mathrm{Aut}(V_+)$).

[13] For example, this is part of Theorem 8.5 of [57], with the Cartan decomposition given in Lemma 8.6 of that book and the proof. It also follows from the conjunction of Theorem III.2.1 and Theorem III.3.1 in [32] (this conjunction is, roughly, Satake's Theorem 8.5).

matrix cases, with $\mathbb{D}$ one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, we have that $(\mathrm{Aut}V_+)_0$ is the group generated by $X \mapsto AXA^\dagger$, where A is a nonsingular matrix with entries in $\mathbb{D}$. While in general $A^\dagger \neq A^{-1}$, equality does hold when A is respectively orthogonal, unitary, or quaternionic-unitary; this exemplifies the general fact that for compact groups, representations are isomorphic to their duals. For a general argument, one could for example use the fact (see e.g. [32]) that the interior of the positive cone of a simple EJA is a *symmetric cone*, meaning a Riemannian symmetric space, $\mathrm{Aut}V_+/K$, which is a product $\mathbb{R} \times (\mathrm{Aut}_0(V_+)/K)$ of the euclidean factor $\mathbb{R}$ with the irreducible noncompact symmetric space $\mathrm{Aut}_0(V_+)/K$, i.e. G^s/K, so that V_+° is foliated by copies of G^s/K, which are in fact level sets of the Jordan determinant of V, since G^s's action on V preserves the Jordan determinant. Then one can see directly that one can identify the traceless subspace $\mathfrak{p}_o$ of $\mathfrak{p}$ with the tangent space to any constant-determinant copy of G^s/K at its intersection with $\mathbb{R}e$, in particular with the tangent space to such a submanifold at $e/\mathrm{rank}V = c(\Omega)$, and the action of K by restriction on the unit-trace subspace, i.e. $\mathrm{aff}(\Omega) \equiv \mathfrak{p}_0 + e/\mathrm{rank}V$, is precisely its action on the tangent space, i.e. the isotropy action on $\mathfrak{p}_0$, if one views $\mathrm{aff}(\Omega)$ as a linear space by identifying $c(\Omega)$ with zero.

Recall that the normalized state space, $\Omega := V_+ \cap \{x \in V | \mathrm{tr}x = 1\}$, is also $\mathrm{conv}(K.\omega)$ for any extremal $\omega \in \Omega$ (from strong symmetry). Everything in the affine plane $\{x \in V | \mathrm{tr}x = 1\}$ has the form $c(\Omega) \oplus \omega_0$, where $c(\Omega) = e/\mathrm{rank}\, V$, the barycenter of Ω, is the unit-trace element of the fixed-point space $\mathbb{R}e$, and $\omega_0 \in \mathfrak{p}_0$.

The maximal abelian subspaces of $V \simeq \mathfrak{p}$ (viewed as subspaces of $\mathfrak{g} = \mathfrak{lie}(\mathrm{Aut}V_+)$) are of the form $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathbb{R}e$, where $\mathfrak{a}_0$ are the maximal abelian subspaces of $\mathfrak{p}_0$. In, for example, [32], Proposition VI.3.3, and the discussion preceding it, the Peirce decomposition $\oplus_{i,j=1}^r V_{ij}$ of a simple EJA of rank r is used, and it is observed that $\mathfrak{a} = \oplus_i V_{ii}$ where $V_{ii} = \mathbb{R}c_i$, and the c_i, $i \in \{1, \ldots, r\}$ are a Jordan frame. With Ω defined as usual as the normalized state space embedded in $V = \mathfrak{p}$, with the Jordan unit e as order unit, we have that $\mathfrak{a} \cap \Omega$ is the simplex generated by the c_i, which is affinely isomorphic to $\mathfrak{a}_0 \cap \Omega_0$, where $\Omega_0 := \Omega - e/\mathrm{tr}e = \Omega - \frac{e}{r}$ is the translation of the normalized Jordan state space into $\mathfrak{a}_0$. Moreover, by Proposition 6.2 $\mathfrak{a}_0 \cap \Omega_0 = \pi(\Omega_0)$, since by Propositions 3.2 and 3.4 and Lemma 5.2, Ω_0 is regular. ■

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